MATH 441: Practice Quiz 2 — Solutions

Problem 1 (12 points)

Find the general solution of each ODE, with any method of your choosing.

a) $y''(t) + y'(t) - 6y(t) = e^{3t}$.

b) $y''(t) - 9y(t) = \cosh(t)$.

Solutions: **a)** The characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, so the homogeneous equation has the fundamental set $\{e^{2t}, e^{-3t}\}$. Since 3 is not a root of p, we try $y_p(t) = Ce^{3t}$. Then

$$y_p''(t) + y_p'(t) - 6y_p(t) = (9 + 3 - 6)Ce^{3t} = 6Ce^{3t},$$

which equals e^{3t} if and only if $C = \frac{1}{6}$. By Theorem 3.3, the general solution is

$$y(t) = c_1e^{2t} + c_2e^{-3t} + \frac{1}{6}e^{3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

b) The characteristic polynomial is $p(\lambda) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3)$, so the homogeneous equation has the fundamental set $\{e^{3t}, e^{-3t}\}$. The right-hand side is $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$, and neither 1 nor -1 is a root of p. Since $\cosh'' = \cosh$, we try $y_p(t) = A\cosh(t)$. Then

$$y_p''(t) - 9y_p(t) = A\cosh(t) - 9A\cosh(t) = -8A\cosh(t),$$

which equals $\cosh(t)$ if and only if $A = -\frac{1}{8}$. (Equivalently, the trial form $Ae^t + Be^{-t}$ gives $A = B = -\frac{1}{16}$.) By Theorem 3.3, the general solution is

$$y(t) = c_1e^{3t} + c_2e^{-3t} - \frac{1}{8}\cosh(t), \qquad c_1, c_2 \in \mathbb{R}.$$

Problem 2 (12 points)

Use the **Laplace transform** to solve the initial value problem

$$y''(t) - 6y'(t) + 9y(t) = e^{3t}, \qquad y(0) = 2, \quad y'(0) = 1.$$

Solutions: Let $Y = \mathcal{L}[y]$. By rule f) of Theorem 4.2, $y'(t) \circ\!-\!\bullet\, sY(s) - 2$ and $y''(t) \circ\!-\!\bullet\, s^2Y(s) - 2s - 1$, and the table gives $e^{3t} \circ\!-\!\bullet\, \frac{1}{s-3}$. Transforming the ODE gives

$$s^2Y(s) - 2s - 1 - 6\,(sY(s) - 2) + 9Y(s) = \frac{1}{s-3} \quad \Longrightarrow \quad (s-3)^2Y(s) = 2s - 11 + \frac{1}{s-3}.$$

Since $2s - 11 = 2(s-3) - 5$,

$$Y(s) = \frac{2s-11}{(s-3)^2} + \frac{1}{(s-3)^3} = \frac{2}{s-3} - \frac{5}{(s-3)^2} + \frac{1}{(s-3)^3}.$$

The table entry $t^n e^{\lambda t} \circ\!-\!\bullet\, \frac{n!}{(s-\lambda)^{n+1}}$ with $\lambda = 3$ and $n = 0, 1, 2$ gives

$$y(t) = 2e^{3t} - 5te^{3t} + \frac{t^2}{2}e^{3t} = \left(2 - 5t + \frac{t^2}{2}\right)e^{3t}.$$

(Check: $y(0) = 2$ and $y'(0) = 3 \cdot 2 - 5 = 1$.)

Problem 3 (13 points)

Let $\omega > 0$. Use the **Laplace transform** to solve the initial value problem

$$y''(t) + \omega^2 y(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < \pi, \\ 0 & \text{if } t \geqslant \pi, \end{cases} \qquad y(0) = 0, \quad y'(0) = 1.$$

a) Write the right-hand side in terms of the Heaviside function H, and compute the Laplace transform $Y(s)$ of the solution.

b) Find the solution $y(t)$ for every $t \geqslant 0$.

Solutions: **a)** The right-hand side is $H(t) - H(t-\pi)$, and $H(t) - H(t-\pi) \circ\!-\!\bullet\, \frac{1}{s} - \frac{e^{-\pi s}}{s}$. With $y''(t) \circ\!-\!\bullet\, s^2Y(s) - 1$, the transformed ODE reads

$$s^2Y(s) - 1 + \omega^2 Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} \quad \Longrightarrow \quad Y(s) = \frac{1}{s^2+\omega^2} + R(s) - e^{-\pi s}R(s),$$

where $R(s) = \frac{1}{s(s^2+\omega^2)}$.

b) Partial fractions give

$$R(s) = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2+\omega^2}\right) \bullet\!-\!\circ\, \frac{1}{\omega^2}\,(1 - \cos(\omega t)) = r(t).$$

(Indeed, $\frac{1}{s} - \frac{s}{s^2+\omega^2} = \frac{\omega^2}{s(s^2+\omega^2)}$.) By rule c) of Theorem 4.2, $e^{-\pi s}R(s) \bullet\!-\!\circ\, H(t-\pi)r(t-\pi)$, and $\frac{1}{s^2+\omega^2} \bullet\!-\!\circ\, \frac{\sin(\omega t)}{\omega}$. Therefore,

$$y(t) = \frac{\sin(\omega t)}{\omega} + \frac{1 - \cos(\omega t)}{\omega^2} - H(t-\pi)\frac{1 - \cos(\omega(t-\pi))}{\omega^2},$$

that is,

$$y(t) = \begin{cases} \dfrac{\sin(\omega t)}{\omega} + \dfrac{1-\cos(\omega t)}{\omega^2} & \text{if } 0 \leqslant t < \pi, \\ \dfrac{\sin(\omega t)}{\omega} + \dfrac{\cos(\omega(t-\pi)) - \cos(\omega t)}{\omega^2} & \text{if } t \geqslant \pi. \end{cases}$$

(Check: for $0 < t < \pi$, $y''(t)+\omega^2 y(t) = 1$, because $\sin(\omega t)$ and $\cos(\omega t)$ solve the homogeneous equation; for $t > \pi$, every term solves the homogeneous equation, so $y''(t)+\omega^2 y(t) = 0$. Since $r(0) = r'(0) = 0$, both y and y' are continuous at $t = \pi$.)

Problem 4 (13 points)

Let $\omega > 0$, and consider the initial value problem

$$y''(t) + \omega^2 y(t) = \cos(\omega t), \qquad y(0) = 0, \quad y'(0) = 0.$$

a) Find the transfer function and the Green's function of the ODE.

b) Use a) to write the solution of the initial value problem as a convolution integral. Do not evaluate the integral in this part.

c) Evaluate the convolution integral in b).

Solutions: **a)** The characteristic polynomial is $p(s) = s^2 + \omega^2$, so the transfer function is

$$G(s) = \frac{1}{s^2+\omega^2},$$

and the table entry $\sin(\omega t) \circ\!-\!\bullet\, \frac{\omega}{s^2+\omega^2}$ gives the Green's function

$$g(t) = \frac{\sin(\omega t)}{\omega}.$$

b) Both initial values vanish, so $Y(s) = G(s)F(s)$ with $F = \mathcal{L}[\cos(\omega t)]$. By Theorem 4.3, $y = g * f$ with $f(t) = \cos(\omega t)$, that is,

$$y(t) = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\cos(\omega\tau)\,d\tau, \qquad t \geqslant 0.$$

c) The identity $\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$ with $a = \omega(t-\tau)$ and $b = \omega\tau$ gives

$$\sin(\omega(t-\tau))\cos(\omega\tau) = \frac{1}{2}\left(\sin(\omega t) + \sin(\omega(t-2\tau))\right).$$

For fixed t, we integrate in τ:

$$\int_0^t \sin(\omega t)\,d\tau = t\sin(\omega t),$$

$$\int_0^t \sin(\omega(t-2\tau))\,d\tau = \left[\frac{\cos(\omega(t-2\tau))}{2\omega}\right]_{\tau=0}^{\tau=t} = \frac{\cos(\omega t) - \cos(\omega t)}{2\omega} = 0.$$

Therefore,

$$y(t) = \frac{1}{2\omega} t \sin(\omega t).$$

(Check with the table: $Y(s) = \frac{s}{(s^2+\omega^2)^2}$, and $t\sin(\omega t) \circ\!-\!\bullet\, \frac{2\omega s}{(s^2+\omega^2)^2} = 2\omega Y(s)$, in agreement with the formula above.)