MATH 441: Practice Quiz 2

Duration: 50 minutes

Please note:

- The practice quiz contains four problems, for a total of 50 points.
- A Notes sheet with Theorem 4.2, a table of Laplace transforms, and Theorem 4.3 is provided at the end of the quiz.
- I suggest you spend about one minute per point.

Problem 1 (12 points)

Find the general solution of each ODE, with any method of your choosing.

a) $y''(t) + y'(t) - 6y(t) = e^{3t}$.

b) $y''(t) - 9y(t) = \cosh(t)$.

Problem 2 (12 points)

Use the **Laplace transform** to solve the initial value problem

$$y''(t) - 6y'(t) + 9y(t) = e^{3t}, \qquad y(0) = 2, \quad y'(0) = 1.$$

Problem 3 (13 points)

Let $\omega > 0$. Use the **Laplace transform** to solve the initial value problem

$$y''(t) + \omega^2 y(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < \pi, \\ 0 & \text{if } t \geqslant \pi, \end{cases} \qquad y(0) = 0, \quad y'(0) = 1.$$

a) Write the right-hand side in terms of the Heaviside function H, and compute the Laplace transform $Y(s)$ of the solution.

b) Find the solution $y(t)$ for every $t \geqslant 0$.

Problem 4 (13 points)

Let $\omega > 0$, and consider the initial value problem

$$y''(t) + \omega^2 y(t) = \cos(\omega t), \qquad y(0) = 0, \quad y'(0) = 0.$$

a) Find the transfer function and the Green's function of the ODE.

b) Use a) to write the solution of the initial value problem as a convolution integral. Do not evaluate the integral in this part.

c) Evaluate the convolution integral in b).

Notes sheet

Laplace transform

For a piecewise continuous function f on $[0,\infty)$ of exponential order, the Laplace transform is

$$\mathcal{L}[f](s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt. \qquad \text{(Notation: } f(t) \circ\!-\!\bullet\, F(s).\text{)}$$

Theorem 4.2 (Rules)

Let f and g be piecewise continuous and of exponential order, with $f \circ\!-\!\bullet\, F$ and $g \circ\!-\!\bullet\, G$. Then the following rules hold for every sufficiently large s:

a) $c_1 f(t) + c_2 g(t) \circ\!-\!\bullet\, c_1 F(s) + c_2 G(s)$ for $c_1, c_2 \in \mathbb{R}$ (or $\mathbb{C}$).

b) $e^{\lambda t} f(t) \circ\!-\!\bullet\, F(s-\lambda) \quad (\lambda \in \mathbb{R})$.

c) $\begin{cases} 0, & t < c \\ f(t-c), & t \geqslant c \end{cases} \circ\!-\!\bullet\; e^{-cs} F(s) \quad (c > 0)$.

d) $t^n f(t) \circ\!-\!\bullet\, (-1)^n F^{(n)}(s)$, where $F^{(n)}$ is the nth derivative.

e) $\int_0^t f(\tau)\,d\tau \circ\!-\!\bullet\, \frac{F(s)}{s}$.

f) $f'(t) \circ\!-\!\bullet\, sF(s) - f(0^+)$, provided that f is continuous on $(0,\infty)$ and f' is piecewise continuous. Here $f(0^+) := \lim_{t\to 0^+} f(t)$ is the right-hand limit of f at 0.

In particular, $y'(t) \circ\!-\!\bullet\, sY(s) - y(0)$ and $y''(t) \circ\!-\!\bullet\, s^2 Y(s) - sy(0) - y'(0)$. With the Heaviside function $H(t) = 1$ for $t \geqslant 0$ and $H(t) = 0$ otherwise, rule c) reads $H(t-c)f(t-c) \circ\!-\!\bullet\, e^{-cs}F(s)$, so that $H(t-c) \circ\!-\!\bullet\, \frac{e^{-cs}}{s}$.

Table of Laplace transforms

Here $n \in \mathbb{N}$ and $\omega \in \mathbb{R}$.

$f(t)$	$F(s)$	Valid for
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{\lambda t}$, $\lambda \in \mathbb{C}$	$\dfrac{n!}{(s-\lambda)^{n+1}}$	$s > \operatorname{Re}\lambda$
$\cos(\omega t)$, $\sin(\omega t)$	$\dfrac{s}{s^2+\omega^2}$, $\dfrac{\omega}{s^2+\omega^2}$	$s > 0$
$e^{\lambda t}\cos(\omega t)$, $e^{\lambda t}\sin(\omega t)$, $\lambda \in \mathbb{R}$	$\dfrac{s-\lambda}{(s-\lambda)^2+\omega^2}$, $\dfrac{\omega}{(s-\lambda)^2+\omega^2}$	$s > \lambda$
$t\cos(\omega t)$, $t\sin(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$, $\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$s > 0$

Theorem 4.3 (Convolution rule)

For two functions f and g, the convolution $f * g$ is $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$. Let f and g be piecewise continuous, with $|f(t)| \leqslant Ke^{\alpha t}$ and $|g(t)| \leqslant Ke^{\alpha t}$ for every $t \geqslant 0$, where $K > 0$ and $\alpha \in \mathbb{R}$. Then $f * g$ is well defined and

$$(f * g)(t) \circ\!-\!\bullet\, F(s)G(s) \qquad (s > \alpha),$$

where $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$.