MATH 441: Assignment 6

Instructions: Submit your *handwritten* solutions to my mailbox (Harker Hall, Room 214) by *Wednesday, November 18th, 5 PM.* Typed solutions require an accommodation.

Readings

Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima.

Boyce and DiPrima: Sections 5.1–5.7 and 7.1–7.9

Problem 1

Consider the Airy equation $y''(t) - ty(t) = 0$.

a) Explain why $t_0 = 0$ is an ordinary point, and why Theorem 5.1 of the notes guarantees that every series solution about $t_0 = 0$ converges for every $t \in \mathbb{R}$.

b) Substitute $y = \sum_{k=0}^{\infty} c_k t^k$ and show that $c_2 = 0$ and $(k+2)(k+1)c_{k+2} = c_{k-1}$ for $k \geq 1$.

c) Show that $c_{3m+2} = 0$ for every $m \geq 0$, and find closed-form expressions for c_{3m} and c_{3m+1} in terms of c_0 and c_1. Let y_1 and y_2 be the series solutions with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, respectively. Write both up to the term in t^7.

d) Use the ratio test on the nonzero terms to show that y_1 and y_2 converge on $\mathbb{R}$.

e) Let y be a solution with $y(0) > 0$ and $y'(0) \geq 0$. Show that $y(t) > 0$, $y'(t) > 0$, and $y''(t) > 0$ for every $t > 0$. **Hint:** Write $y'(t) = y'(0) + \int_0^t sy(s)\, ds$.

Problem 2

Consider the ODE $(1 - t)y''(t) + ty'(t) - y(t) = 0$.

a) Show that $t_0 = 0$ is an ordinary point, and that Theorem 5.1 of the notes guarantees that every series solution about $t_0 = 0$ converges at least for $|t| < 1$.

b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^k$ into the ODE and show that

$$(k + 2)(k + 1)c_{k+2} = k(k + 1)c_{k+1} - (k - 1)c_k \qquad \text{for every } k \geq 0.$$

c) Show by induction that $c_k = c_0/k!$ for every $k \geq 2$. Deduce that the general solution is $y(t) = c_0(e^t - t) + c_1 t$, and check that $e^t - t$ and t solve the ODE.

d) The two series in c) converge for every $t \in \mathbb{R}$, while a) guarantees convergence only for $|t| < 1$. Explain why this does not contradict Theorem 5.1 of the notes.

Problem 3

Consider the Bessel equation of order 0,

$$ty''(t) + y'(t) + ty(t) = 0, \qquad t > 0.$$

a) Show that $t_0 = 0$ is a regular singular point, and that the indicial equation has the double root $r_1 = r_2 = 0$. Which case of Theorem 5.2 of the notes applies?

b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^{k+r}$ into the ODE and show that

$$r^2 c_0 = 0, \qquad (r+1)^2 c_1 = 0, \qquad (k+r)^2 c_k = -c_{k-2} \quad \text{for every } k \geq 2.$$

c) Let $r = 0$ and $c_0 = 1$. Show that the odd coefficients vanish and that $c_{2m} = (-1)^m / (4^m (m!)^2)$ for $m \geq 0$. The resulting solution is the Bessel function of the first kind of order 0:

$$y_1(t) = J_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{t}{2}\right)^{2m}.$$

d) By Theorem 5.2(c), a second solution is $y_2(t) = y_1(t) \ln t + v(t)$ with $v(t) = \sum_{k=1}^{\infty} d_k t^k$. Substitute y_2 into the ODE and show that v satisfies

$$tv''(t) + v'(t) + tv(t) = -2y_1'(t).$$

Deduce that $d_1 = d_3 = 0$, $d_2 = \frac{1}{4}$, and $d_4 = -\frac{3}{128}$, so that

$$y_2(t) = J_0(t) \ln t + \frac{t^2}{4} - \frac{3t^4}{128} + \cdots$$

e) Which solutions on $(0, \infty)$ are bounded as $t \to 0^+$?

Problem 4

Consider the ODE

$$ty''(t) + 2y'(t) + ty(t) = 0, \qquad t > 0.$$

a) Show that $t_0 = 0$ is a regular singular point, and that the roots of the indicial equation are $r_1 = 0$ and $r_2 = -1$. Which case of Theorem 5.2 of the notes applies?

b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^{k+r}$ into the ODE and show that

$$r(r+1)c_0 = 0, \qquad (r+1)(r+2)c_1 = 0, \qquad (k+r)(k+r+1)c_k = -c_{k-2} \quad \text{for every } k \geq 2.$$

c) Let $r = r_2 = -1$. Show that c_0 and c_1 are free, find c_{2m} and c_{2m+1} in closed form, and deduce that the general solution on $(0, \infty)$ is

$$y(t) = \frac{c_0 \cos t + c_1 \sin t}{t}.$$

d) Show that y_1 of Theorem 5.2(a) is $\sin t/t$, and that $y_2 = \cos t/t$ has the form of Theorem 5.2(d) with $\kappa = 0$. Compare with Problem 3: why is there no logarithm?

e) Which solutions on $(0, \infty)$ have a finite limit as $t \to 0^+$? Compute the limit.

Problem 5

Let $\boldsymbol{A} = \left(\begin{smallmatrix} 1 & 1 \\ 4 & 1 \end{smallmatrix}\right)$.

a) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$, and the general solution of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$.

b) Solve the initial value problem with $\boldsymbol{y}(0) = (1, 0)^\top$.

c) Compute $e^{\boldsymbol{A}t}$ by diagonalization (case (1) of Section 6.3.2 of the notes). Check that $e^{\boldsymbol{A}\cdot 0} = \boldsymbol{I}$, and that the first column of $e^{\boldsymbol{A}t}$ is the solution found in b).

d) Compute $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$, invert each entry, and compare with c).

e) Let $\boldsymbol{f}(t) = (0, 1)^\top$. Solve $\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{f}(t)$ with $\boldsymbol{y}(0) = \boldsymbol{0}$ using Corollary 6.1 of the notes, and check your answer with the Laplace transform (Section 6.3.3 of the notes).

Problem 6

Consider the matrices

$$\boldsymbol{A}_1 = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.$$

a) Show that $\boldsymbol{A}_1$ has the eigenvalues $\pm i$, find an eigenvector for $\lambda = i$, and write a real fundamental set $\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ of $\boldsymbol{y}' = \boldsymbol{A}_1\boldsymbol{y}$ (Section 6.3.1 of the notes).

b) Compute $e^{\boldsymbol{A}_1 t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}$ from the fundamental matrix $\boldsymbol{\Phi} = [\boldsymbol{y}_1\ \boldsymbol{y}_2]$ of a).

c) Show that $\boldsymbol{A}_1^2 = -\boldsymbol{I}$, and use the definition of the matrix exponential to show that $e^{\boldsymbol{A}_1 t} = (\cos t)\boldsymbol{I} + (\sin t)\boldsymbol{A}_1$. Compare with b).

d) Show that $\boldsymbol{A}_2$ is not diagonalizable. Compute $\boldsymbol{A}_2^2$, $\boldsymbol{A}_2^3$, and $\boldsymbol{A}_2^4$, guess a formula for $\boldsymbol{A}_2^k$, prove it by induction, and use the definition of the matrix exponential to compute $e^{\boldsymbol{A}_2 t}$. Then solve the initial value problem $\boldsymbol{y}' = \boldsymbol{A}_2\boldsymbol{y}$, $\boldsymbol{y}(0) = (1, 0)^\top$.

Problem 7

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be constant.

a) Apply the Abel–Liouville identity (Proposition 6.3 of the notes) to the fundamental matrix $e^{\boldsymbol{A}t}$ to show that $\det e^{\boldsymbol{A}t} = e^{t\,\mathrm{tr}\,\boldsymbol{A}}$ for every $t \in \mathbb{R}$. In particular, $\det e^{\boldsymbol{A}} = e^{\mathrm{tr}\,\boldsymbol{A}}$. Check this identity on the matrices of Problems 5 and 6.

b) Show that $\left(e^{\boldsymbol{A}}\right)^{\top} = e^{\boldsymbol{A}^{\top}}$. **Hint:** Transpose the partial sums of the series.

c) Suppose that $\boldsymbol{A}$ is skew-symmetric, that is, $\boldsymbol{A}^{\top} = -\boldsymbol{A}$. Show that $\|\boldsymbol{y}(t)\|$ is constant for every solution $\boldsymbol{y}$ of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, and that $e^{\boldsymbol{A}t}$ is an orthogonal matrix, that is, $\left(e^{\boldsymbol{A}t}\right)^{\top} e^{\boldsymbol{A}t} = \boldsymbol{I}$. **Hint:** Differentiate $\|\boldsymbol{y}(t)\|^2 = \boldsymbol{y}(t)^{\top}\boldsymbol{y}(t)$, and use b) and Proposition 6.5 of the notes. Check the second claim on $\boldsymbol{A} = \tau\boldsymbol{J}$, where $\boldsymbol{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\tau \in \mathbb{R}$.

d) Now let a_1, a_0, and f be continuous on an open interval I, let $\{y_1, y_2\}$ be a fundamental set of solutions of $y'' + a_1(t)y' + a_0(t)y = 0$ with Wronskian W, and let

$$\boldsymbol{\Phi}(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

be the corresponding fundamental matrix of the companion system (Example 6.2 of the notes). Compute $\boldsymbol{\Phi}(s)^{-1}$ and show that the $(1,2)$ entry of $\boldsymbol{R}(t,s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}$ is

$$\frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(s)}.$$

Then write Duhamel's formula (Theorem 6.3 of the notes) for the companion system of $y'' + a_1(t)y' + a_0(t)y = f(t)$ with $\boldsymbol{z}(t_0) = \boldsymbol{0}$, and show that its first component is the particular solution y_p of Section 3.3.2 of the notes.

Problem 8

Consider the coupled second-order system

$$y_1''(t) = -6y_1(t) + 4y_2(t), \qquad y_2''(t) = 2y_1(t) - 4y_2(t),$$

with $y_1(0) = 3$, $y_2(0) = 0$, and $y_1'(0) = y_2'(0) = 0$.

a) Solve the initial value problem with the Laplace transform.

b) Write the system as $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ for $\boldsymbol{y} = (y_1, y_2, y_1', y_2')^{\top}$, with $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$. Show that $\boldsymbol{y}(t) = e^{\lambda t}(\boldsymbol{u}, \lambda\boldsymbol{u})^{\top}$ with $\boldsymbol{u} \in \mathbb{R}^2$ nonzero solves the system if and only if

$$\boldsymbol{K}\boldsymbol{u} = \lambda^2\boldsymbol{u}, \qquad \text{where } \boldsymbol{K} = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{K}$, deduce the four eigenvalues of $\boldsymbol{A}$ and their eigenvectors, and write the general real solution of the first-order system. Then solve the initial value problem again and compare with a).

c) Is the motion periodic? If so, find its period.