MATH 441: Solutions to Assignment 5

Problem 1

Let $\omega, c \in \mathbb{R}$. Compute the Laplace transform of each function, and state the values of s for which it is defined.

a) $\sinh(\omega t)$ and $\cosh(\omega t)$. **Hint:** Write both functions with exponentials and use Example 4.1 of the notes.

b) te^{ct} and t^2e^{ct}. Use rule d) of Theorem 4.2 of the notes, and check your answers with rule b).

c) $e^{-2t}t\cos(3t)$.

d) The function f defined by $f(t) = t$ for $0 \leq t < 1$ and $f(t) = 0$ for $t \geq 1$. **Hint:** Write f with Heaviside functions, and write $t = (t-1) + 1$.

e) $t^2H(t-1)$, where H is the Heaviside function.

Solutions: **a)** We have $\sinh(\omega t) = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$ and $\cosh(\omega t) = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$. By Example 4.1 and linearity, for $s > |\omega|$ (so that $s > \omega$ and $s > -\omega$),

$$\sinh(\omega t) \circ\!-\!\bullet\ \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{\omega}{s^2-\omega^2},$$
$$\cosh(\omega t) \circ\!-\!\bullet\ \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}.$$

b) Let $F(s) = \frac{1}{s-c}$, the transform of e^{ct} for $s > c$. Rule d) with $n = 1$ and $n = 2$ gives, for $s > c$,

$$te^{ct} \circ\!-\!\bullet\ -F'(s) = \frac{1}{(s-c)^2}, \qquad t^2e^{ct} \circ\!-\!\bullet\ F''(s) = \frac{2}{(s-c)^3}.$$

Check with rule b): $t \circ\!-\!\bullet\ \frac{1}{s^2}$ and $t^2 \circ\!-\!\bullet\ \frac{2}{s^3}$, so shifting s to $s-c$ gives the same two transforms.

c) The table gives $t\cos(3t) \circ\!-\!\bullet\ \frac{s^2-9}{(s^2+9)^2}$ for $s > 0$. Rule b) with $\lambda = -2$ gives

$$e^{-2t}t\cos(3t) \circ\!-\!\bullet\ \frac{(s+2)^2-9}{\left((s+2)^2+9\right)^2}, \qquad s > -2.$$

d) We have $f(t) = t\,(H(t) - H(t-1)) = t - (t-1)H(t-1) - H(t-1)$. Rule c) gives

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.$$

Since $f(t) = 0$ for $t \geq 1$, the integral $F(s) = \int_0^1 te^{-st}\,dt$ is defined for every $s \in \mathbb{R}$. Integration by parts gives the formula above for every $s \neq 0$, and $F(0) = \int_0^1 t\,dt = \frac{1}{2}$.

e) We write $t^2 = (t-1)^2 + 2(t-1) + 1$, so that $t^2H(t-1) = H(t-1)g(t-1)$ with $g(t) = t^2 + 2t + 1 \circ\!-\!\bullet\, \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$. Rule c) gives

$$t^2H(t-1) \circ\!-\!\bullet\, e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right), \qquad s > 0.$$

Problem 2

Find the inverse Laplace transform of each function.

a) $\dfrac{2s+1}{s^2+4s+13}$.

b) $\dfrac{1}{s^2(s+1)}$.

c) $\dfrac{e^{-\pi s}}{s(s^2+1)}$.

d) $F(s) = \ln\left(\dfrac{s+1}{s-1}\right)$, for $s > 1$. You may assume that F is the Laplace transform of a piecewise continuous function f of exponential order. **Hint:** Use rule d) of Theorem 4.2 of the notes to find $tf(t)$.

Solutions: **a)** We complete the square: $s^2+4s+13 = (s+2)^2+9$ and $2s+1 = 2(s+2)-3$. The table and rule b) give

$$\frac{2s+1}{s^2+4s+13} = 2\frac{s+2}{(s+2)^2+9} - \frac{3}{(s+2)^2+9} \bullet\!-\!\circ\, e^{-2t}\left(2\cos(3t) - \sin(3t)\right).$$

b) Partial fractions give

$$\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \bullet\!-\!\circ\, -1 + t + e^{-t}.$$

c) Partial fractions give $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \bullet\!-\!\circ\, 1 - \cos t$. Rule c) with $c = \pi$ gives

$$\frac{e^{-\pi s}}{s(s^2+1)} \bullet\!-\!\circ\, H(t-\pi)\left(1 - \cos(t-\pi)\right) = H(t-\pi)\left(1+\cos t\right).$$

d) We have $F'(s) = \frac{1}{s+1} - \frac{1}{s-1}$. Rule d) with $n = 1$ and Example 4.1 give

$$tf(t) \circ\!-\!\bullet\, -F'(s) = \frac{1}{s-1} - \frac{1}{s+1} \bullet\!-\!\circ\, e^t - e^{-t} = 2\sinh t.$$

By Lerch's theorem, $tf(t) = 2\sinh t$, so $f(t) = \dfrac{2\sinh t}{t}$ for $t > 0$.

Problem 3

Use the Laplace transform to solve each initial value problem.

a) $y''(t) - 4y'(t) + 4y(t) = te^{2t}$, with $y(0) = 1$ and $y'(0) = 0$.

b) $y'''(t) - y''(t) + y'(t) - y(t) = 0$, with $y(0) = 1$ and $y'(0) = y''(0) = 0$.

c) $y''(t) + 2y'(t) + 2y(t) = g(t)$, with $y(0) = 0$ and $y'(0) = 1$, where

$$g(t) = \begin{cases} 1, & \pi \le t < 2\pi, \\ 0, & 0 \le t < \pi \text{ or } t \ge 2\pi. \end{cases}$$

Solutions: **a)** By Problem 1 b) with $c = 2$, the right-hand side has transform $\frac{1}{(s-2)^2}$. Transforming the ODE,

$$\begin{aligned} & s^2Y - s - 4(sY - 1) + 4Y = \frac{1}{(s-2)^2} \\ \Longrightarrow \quad & Y(s) = \frac{s-4}{(s-2)^2} + \frac{1}{(s-2)^4} = \frac{1}{s-2} - \frac{2}{(s-2)^2} + \frac{1}{(s-2)^4}, \end{aligned}$$

where we wrote $s - 4 = (s - 2) - 2$. The table gives

$$y(t) = e^{2t}\left(1 - 2t + \frac{t^3}{6}\right).$$

b) With the initial values, $\mathcal{L}[y'''] = s^3Y - s^2$, $\mathcal{L}[y''] = s^2Y - s$, and $\mathcal{L}[y'] = sY - 1$. Transforming the ODE gives $(s^3-s^2+s-1)Y = s^2-s+1$. Since $s^3-s^2+s-1 = (s-1)(s^2+1)$, partial fractions give

$$Y(s) = \frac{s^2 - s + 1}{(s-1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}\right) \bullet\!-\!\circ\, y(t) = \frac{1}{2}\left(e^t + \cos t - \sin t\right).$$

c) We have $g(t) = H(t - \pi) - H(t - 2\pi)$. Transforming the ODE,

$$\begin{aligned} & (s^2 + 2s + 2)Y - 1 = \frac{e^{-\pi s} - e^{-2\pi s}}{s} \\ \Longrightarrow \quad & Y(s) = \frac{1}{(s+1)^2+1} + \left(e^{-\pi s} - e^{-2\pi s}\right) R(s), \qquad R(s) = \frac{1}{s\left((s+1)^2+1\right)}. \end{aligned}$$

Partial fractions give

$$R(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right) \bullet\!-\!\circ\, r(t) = \frac{1}{2}\left(1 - e^{-t}\cos t - e^{-t}\sin t\right).$$

By rule c),

$$y(t) = e^{-t}\sin t + H(t-\pi)r(t-\pi) - H(t-2\pi)r(t-2\pi).$$

Since $\cos(t - \pi) = -\cos t$, $\sin(t - \pi) = -\sin t$, and cos and sin are 2π-periodic,

$$y(t) = \begin{cases} e^{-t}\sin t, & 0 \le t < \pi, \\ e^{-t}\sin t + \frac{1}{2} + \frac{1}{2}e^{\pi - t}(\cos t + \sin t), & \pi \le t < 2\pi, \\ e^{-t}\sin t + \frac{1}{2}\left(e^{\pi-t} + e^{2\pi-t}\right)(\cos t + \sin t), & t \ge 2\pi. \end{cases}$$

Problem 4

Let $k \in \mathbb{R}$ and $T > 0$. A mass on a spring starts at equilibrium with unit velocity and receives an impulse of magnitude k at time T:

$$y''(t) + y(t) = k\delta_0(t - T), \qquad y(0) = 0, \quad y'(0) = 1.$$

a) Rule c) of Theorem 4.2 of the notes and $\delta_0(t) \circ\!-\!\bullet\, 1$ give, formally, $\delta_0(t-T) \circ\!-\!\bullet\, e^{-Ts}$. Use this rule to show that $y(t) = \sin t + kH(t-T)\sin(t-T)$. Show that y is continuous at $t = T$ and that y' jumps by k at $t = T$.

b) Find all pairs (T, k) such that $y(t) = 0$ for every $t \geq T$.

c) Let $\varepsilon > 0$, and let y_ε solve the same initial value problem with $k\delta_0(t - T)$ replaced by $k\delta_\varepsilon(t - T) = \frac{k}{\varepsilon}\left(H(t - T) - H(t - T - \varepsilon)\right)$. Find $y_\varepsilon(t)$ for $t \geq T + \varepsilon$, and show that $\lim_{\varepsilon \to 0} y_\varepsilon(t) = y(t)$ for every $t > T$.

Solutions: **a)** Transforming the ODE,

$$s^2Y - 1 + Y = ke^{-Ts} \quad \Longrightarrow \quad Y(s) = \frac{1}{s^2+1} + ke^{-Ts}\frac{1}{s^2+1}.$$

Since $\frac{1}{s^2+1} \bullet\!-\!\circ\, \sin t$, rule c) gives $y(t) = \sin t + kH(t - T)\sin(t - T)$. The second term vanishes for $t < T$ and tends to $k \sin 0 = 0$ as $t \to T^+$, so y is continuous at T. Moreover, $y'(t) = \cos t$ for $t < T$ and $y'(t) = \cos t + k\cos(t - T)$ for $t > T$, so

$$\lim_{t \to T^+} y'(t) - \lim_{t \to T^-} y'(t) = k \cos 0 = k.$$

b) For $t \geq T$,

$$y(t) = \sin t + k\left(\sin t \cos T - \cos t \sin T\right) = (1 + k\cos T)\sin t - k \sin T \cos t.$$

The functions sin and cos are linearly independent on every interval, so $y(t) = 0$ for every $t \geq T$ if and only if $k \sin T = 0$ and $1 + k\cos T = 0$. The second equation forces $k \neq 0$, so the first gives $\sin T = 0$, that is, $T = n\pi$ with $n \in \{1, 2, \ldots\}$. Then $\cos T = (-1)^n$ and $k = (-1)^{n+1}$. The pairs are

$$(T, k) = \left(n\pi, (-1)^{n+1}\right), \qquad n = 1, 2, \ldots$$

At these times the mass passes through equilibrium with velocity $y'(T^-) = \cos(n\pi) = (-1)^n$, and by a) the impulse changes the velocity by $k = -y'(T^-)$, which brings the mass to rest.

c) The forcing term has transform $\frac{k}{\varepsilon}\frac{e^{-Ts} - e^{-(T+\varepsilon)s}}{s}$, so

$$Y_\varepsilon(s) = \frac{1}{s^2+1} + \frac{k}{\varepsilon}\left(e^{-Ts} - e^{-(T+\varepsilon)s}\right)\frac{1}{s(s^2+1)}.$$

By Problem 2 c), $\frac{1}{s(s^2+1)} \bullet\!-\!\circ\, 1-\cos t$, and rule c) gives

$$y_\varepsilon(t) = \sin t + \frac{k}{\varepsilon}\left[H(t-T)\,(1-\cos(t-T)) - H(t-T-\varepsilon)\,(1-\cos(t-T-\varepsilon))\right].$$

For $t \geq T+\varepsilon$, both Heaviside functions equal 1 and

$$y_\varepsilon(t) = \sin t + k\,\frac{\cos(t-T-\varepsilon)-\cos(t-T)}{\varepsilon}.$$

Fix $t > T$. For every $\varepsilon < t-T$ this formula holds, and the quotient is a difference quotient of $u \mapsto \cos(t-T-u)$ at $u=0$. Its limit as $\varepsilon \to 0$ is the derivative $\sin(t-T)$, so $\lim_{\varepsilon\to 0} y_\varepsilon(t) = \sin t + k\sin(t-T) = y(t)$.

Problem 5

a) Compute the convolution $(\sin * \cos)(t)$ directly from the definition. Then check your answer with Theorem 4.3 of the notes.

b) Use the Laplace transform to solve each integral equation. You may assume that the solution y is continuous and of exponential order.

(1) $y(t) + \int_0^t (t-\tau) y(\tau)\, d\tau = 1$;

(2) $y(t) = t + \int_0^t \sin(t-\tau) y(\tau)\, d\tau$.

c) Let $m, n \in \mathbb{N}$. Use Theorem 4.3 of the notes to show that

$$\int_0^t \tau^m (t-\tau)^n \, d\tau = \frac{m!\, n!}{(m+n+1)!} t^{m+n+1} \qquad \forall\, t \geq 0.$$

Deduce that $\int_0^1 x^m (1-x)^n \, dx = \frac{m!\, n!}{(m+n+1)!}$.

Solutions: **a)** With $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$,

$$\begin{aligned}(\sin * \cos)(t) &= \int_0^t \sin(t-\tau)\cos\tau \, d\tau = \frac{1}{2}\int_0^t (\sin t + \sin(t-2\tau))\, d\tau \\ &= \frac{t}{2}\sin t + \frac{1}{4}\left[\cos(t-2\tau)\right]_0^t = \frac{t}{2}\sin t,\end{aligned}$$

because $\cos(-t) = \cos t$. Check: by Theorem 4.3, $(\sin * \cos)(t) \circ\!-\!\bullet \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} = \frac{1}{2} \cdot \frac{2s}{(s^2+1)^2}$, and the table gives $\frac{2s}{(s^2+1)^2} \bullet\!-\!\circ\, t \sin t$.

b) (1) The integral is $(t * y)(t)$, so Theorem 4.3 gives

$$Y + \frac{Y}{s^2} = \frac{1}{s} \quad \Longrightarrow \quad Y(s) = \frac{s}{s^2+1} \quad \Longrightarrow \quad y(t) = \cos t.$$

(2) The integral is $(\sin * y)(t)$, so

$$Y = \frac{1}{s^2} + \frac{Y}{s^2+1} \quad \Longrightarrow \quad Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \quad \Longrightarrow \quad y(t) = t + \frac{t^3}{6}.$$

c) The left-hand side equals $(t^m * t^n)(t)$. For every $\alpha > 0$, the functions $t^m e^{-\alpha t}$ and $t^n e^{-\alpha t}$ are bounded on $[0, \infty)$, so Theorem 4.3 applies and gives

$$(t^m * t^n)(t) \circ\!-\!\bullet \frac{m!}{s^{m+1}} \cdot \frac{n!}{s^{n+1}} = \frac{m!\, n!}{(m+n+1)!} \cdot \frac{(m+n+1)!}{s^{m+n+2}} \bullet\!-\!\circ \frac{m!\, n!}{(m+n+1)!} t^{m+n+1}.$$

Both sides are continuous on $[0, \infty)$, so Lerch's theorem gives the identity for every $t \geq 0$. Setting $t = 1$ gives $\int_0^1 x^m (1-x)^n \, dx = \frac{m!\, n!}{(m+n+1)!}$.

Problem 6

Consider the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

a) Find the transfer function and the Green's function of the ODE.

b) Let f be continuous and of exponential order. Write the solution of the initial value problem as a convolution integral.

c) Let $f(t) = e^{-t}$. Evaluate the integral in b), and check your answer by computing $Y(s)$ and inverting it.

d) Let $f(t) = e^{-t}$ again, but now impose $y(0) = 1$ and $y'(0) = -1$. Use equation (15) of the notes to show that the solution is the sum of the solution of the homogeneous equation with these initial values and the solution found in c). Find the solution.

Solutions: **a)** The characteristic polynomial is $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$. The transfer function is $G(s) = \frac{1}{(s+1)^2+4}$, and the table gives the Green's function

$$g(t) = \frac{1}{2} e^{-t} \sin(2t).$$

b) By Section 4.6 of the notes,

$$y(t) = (g * f)(t) = \frac{1}{2} \int_0^t e^{-(t-\tau)} \sin\left(2(t-\tau)\right) f(\tau)\, d\tau.$$

c) With $f(\tau) = e^{-\tau}$, we have $e^{-(t-\tau)} e^{-\tau} = e^{-t}$, so

$$y(t) = \frac{1}{2} e^{-t} \int_0^t \sin\left(2(t-\tau)\right) d\tau = \frac{1}{2} e^{-t} \left[\frac{\cos\left(2(t-\tau)\right)}{2} \right]_0^t = \frac{1}{4} e^{-t} \left(1 - \cos(2t)\right).$$

Check: $Y(s) = G(s)F(s) = \frac{1}{(s+1)((s+1)^2+4)}$. Partial fractions give $\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \bullet\!-\!\circ\, \frac{1}{4}\left(1 - \cos(2t)\right)$, and rule b) with $\lambda = -1$ gives the same y.

d) Equation (15) of the notes with $a_1 = 2$, $y_0 = 1$, and $y_0' = -1$ gives

$$Y(s) = \frac{(s+2) - 1}{p(s)} + \frac{F(s)}{p(s)} = \frac{s+1}{(s+1)^2 + 4} + G(s)F(s).$$

By equation (15) with $F = 0$, the first term is the transform of the solution y_h of the homogeneous equation with $y_h(0) = 1$ and $y_h'(0) = -1$. The second term is the transform of the solution found in c). By Lerch's theorem, y is the sum of these two functions. The table gives $y_h(t) = e^{-t}\cos(2t)$, so

$$y(t) = e^{-t}\cos(2t) + \frac{1}{4} e^{-t}\left(1 - \cos(2t)\right) = e^{-t}\left(\frac{1}{4} + \frac{3}{4}\cos(2t)\right).$$

Problem 7

Let f be piecewise continuous and of exponential order α, with $f(t) \circ\!-\!\bullet\, F(s)$.

a) Show that $\lim_{s\to\infty} F(s) = 0$. **Hint:** Split the integral at $t = M$, as in the proof of Theorem 4.1 of the notes. Conclude that no piecewise continuous function of exponential order has Laplace transform 1.

b) Suppose, in addition, that f is continuous on $(0, \infty)$ and that f' is piecewise continuous and of exponential order. Prove the *initial value theorem*:

$$\lim_{s\to\infty} sF(s) = f(0^+).$$

Check this formula for $f(t) = \cos(\omega t)$.

c) Suppose that $F(s) = \dfrac{2s^2+1}{(s+1)(s^2+1)}$ is the Laplace transform of a function f that satisfies the hypotheses of b). Find $f(0^+)$ without computing f. Then compute f and check your answer.

Solutions: **a)** Let K, α, and M be as in Theorem 4.1. A piecewise continuous function is bounded on $[0, M]$, so there exists $B \geq 0$ with $|f(t)| \leq B$ for $0 \leq t \leq M$. For $s > \max(\alpha, 0)$,

$$|F(s)| \leq \int_0^M e^{-st}|f(t)|\,dt + \int_M^\infty e^{-st}Ke^{\alpha t}\,dt \leq B\int_0^\infty e^{-st}\,dt + K\int_0^\infty e^{-(s-\alpha)t}\,dt = \frac{B}{s} + \frac{K}{s-\alpha}.$$

The right-hand side tends to 0 as $s \to \infty$, so $\lim_{s\to\infty} F(s) = 0$. The constant function 1 does not tend to 0, so it is not the Laplace transform of a piecewise continuous function of exponential order.

b) By rule f), $\mathcal{L}[f'](s) = sF(s) - f(0^+)$. Since f' is piecewise continuous and of exponential order, a) applied to f' gives $\mathcal{L}[f'](s) \to 0$ as $s \to \infty$. Hence $sF(s) \to f(0^+)$. Check: for $f(t) = \cos(\omega t)$, $sF(s) = \frac{s^2}{s^2+\omega^2} \to 1 = \cos 0$.

c) By b),

$$f(0^+) = \lim_{s\to\infty} \frac{s(2s^2+1)}{(s+1)(s^2+1)} = \lim_{s\to\infty} \frac{2s^3+s}{s^3+s^2+s+1} = 2.$$

Partial fractions give

$$F(s) = \frac{3/2}{s+1} + \frac{1}{2}\cdot\frac{s}{s^2+1} - \frac{1}{2}\cdot\frac{1}{s^2+1} \;\bullet\!-\!\circ\; f(t) = \frac{3}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}\sin t,$$

and indeed $f(0) = \frac{3}{2} + \frac{1}{2} = 2$.

Problem 8

The Bessel function of the first kind of order 0, denoted J_0, solves the initial value problem

$$ty''(t) + y'(t) + ty(t) = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

You may assume that J_0, J_0', and J_0'' are continuous and bounded on $[0, \infty)$. Let $Y = \mathcal{L}[J_0]$.

a) Use rules d) and f) of Theorem 4.2 of the notes to show that $(s^2+1)Y'(s) + sY(s) = 0$ for every sufficiently large s.

b) Solve the ODE in a) to show that $Y(s) = \dfrac{c}{\sqrt{s^2+1}}$ for some constant c.

c) Use Problem 7 b) to show that $c = 1$, that is, $J_0(t) \circ\!-\!\bullet \dfrac{1}{\sqrt{s^2+1}}$.

Solutions: **a)** Write $y = J_0$. Since y, y', and y'' are continuous and bounded, they are of exponential order 0, and rules d) and f) apply. With $y(0) = 1$ and $y'(0) = 0$, rule f) gives $\mathcal{L}[y'] = sY - 1$ and $\mathcal{L}[y''] = s^2Y - s$. Rule d) with $n = 1$ gives

$$\mathcal{L}[ty''] = -\frac{d}{ds}\left(s^2Y - s\right) = -2sY - s^2Y' + 1, \qquad \mathcal{L}[ty] = -Y'.$$

Transforming the ODE,

$$-2sY - s^2Y' + 1 + sY - 1 - Y' = 0, \qquad \text{that is,} \qquad (s^2+1)Y'(s) + sY(s) = 0.$$

b) Dividing by $\sqrt{s^2+1}$,

$$\frac{d}{ds}\left(\sqrt{s^2+1}\,Y(s)\right) = \sqrt{s^2+1}\,Y'(s) + \frac{s}{\sqrt{s^2+1}}Y(s) = \frac{(s^2+1)Y'(s) + sY(s)}{\sqrt{s^2+1}} = 0.$$

Hence $\sqrt{s^2+1}\,Y(s) = c$ for some constant c, that is, $Y(s) = \frac{c}{\sqrt{s^2+1}}$.

c) The function J_0 is continuous, and J_0' is continuous and bounded, so Problem 7 b) applies:

$$1 = J_0(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{cs}{\sqrt{s^2+1}} = c.$$

Therefore $J_0(t) \circ\!-\!\bullet \frac{1}{\sqrt{s^2+1}}$.