MATH 441: Solutions to Assignment 4

Problem 1

Find the general solution of each ODE. In a), also solve the initial value problem.

a) $y''(t) + 2y'(t) + 5y(t) = 0$, with $y(0) = 1$ and $y'(0) = -3$.

b) $y^{(5)}(t) + 3y^{(4)}(t) - y'''(t) - 7y''(t) + 4y(t) = 0$. **Hint:** Verify that the characteristic polynomial equals $(\lambda - 1)^2(\lambda + 2)^2(\lambda + 1)$.

c) The equation *Septimi Gradus* of Leonhard Euler (1743):

$$y^{(7)}(t) + y^{(5)}(t) + y^{(4)}(t) + y'''(t) + y''(t) + y(t) = 0.$$

Hint: Verify that the characteristic polynomial equals $(\lambda+1)(\lambda^2+\lambda+1)(\lambda^2-\lambda+1)^2$.

Solutions: **a)** The characteristic polynomial is $p(\lambda) = \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4$, with roots $-1 \pm 2i$. The general solution is

$$y(t) = e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right), \qquad c_1, c_2 \in \mathbb{R}.$$

Its derivative is $y'(t) = e^{-t}\left((2c_2 - c_1)\cos(2t) - (2c_1 + c_2)\sin(2t)\right)$. The initial conditions give $y(0) = c_1 = 1$ and $y'(0) = 2c_2 - c_1 = -3$, so $c_2 = -1$ and

$$y(t) = e^{-t}\left(\cos(2t) - \sin(2t)\right).$$

b) We have $(\lambda - 1)^2(\lambda + 2)^2 = (\lambda^2 + \lambda - 2)^2 = \lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4$, and multiplying by $\lambda + 1$ gives $\lambda^5 + 3\lambda^4 - \lambda^3 - 7\lambda^2 + 4$, the characteristic polynomial of the ODE. Its roots are 1 (multiplicity 2), -2 (multiplicity 2), and -1 (simple). By Remark 3.1 of the notes, the general solution is

$$y(t) = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-2t} + c_5 e^{-t}, \qquad c_1, \ldots, c_5 \in \mathbb{R}.$$

c) The identities $(\lambda+1)(\lambda^2-\lambda+1) = \lambda^3+1$ and $(\lambda^2+\lambda+1)(\lambda^2-\lambda+1) = \lambda^4+\lambda^2+1$ give

$$(\lambda + 1)(\lambda^2 + \lambda + 1)(\lambda^2 - \lambda + 1)^2 = (\lambda^3 + 1)(\lambda^4 + \lambda^2 + 1) = \lambda^7 + \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + 1,$$

which is the characteristic polynomial of the ODE. The roots are -1 (simple), $\frac{-1\pm i\sqrt{3}}{2}$ (simple), and $\frac{1\pm i\sqrt{3}}{2}$ (each of multiplicity 2). By Remark 3.1 of the notes, the general solution is

$$\begin{aligned} y(t) = c_1 e^{-t} + e^{-t/2}\left(c_2\cos\left(\tfrac{\sqrt{3}}{2}t\right) + c_3\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) \\ + e^{t/2}\left((c_4+c_5t)\cos\left(\tfrac{\sqrt{3}}{2}t\right) + (c_6+c_7t)\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right), \end{aligned}$$

with $c_1, \ldots, c_7 \in \mathbb{R}$.

Problem 2

In this problem, you will prove the claim left as an exercise in Remark 3.1 of the notes, for real roots. Consider the nth-order equation $L[y] = 0$, where

$$L[y](t) = y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$

with real constants $a_0, \ldots, a_{n-1}$, and let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be its characteristic polynomial. Let $\lambda_0 \in \mathbb{R}$ be a root of p of multiplicity m.

a) Show that $L[e^{\lambda t}] = p(\lambda)e^{\lambda t}$ for every $\lambda \in \mathbb{R}$.

b) Differentiate the identity in a) k times with respect to λ to show that

$$L[t^k e^{\lambda t}] = \sum_{j=0}^{k} \binom{k}{j} p^{(j)}(\lambda) t^{k-j} e^{\lambda t}.$$

(You may interchange derivatives with respect to t and λ.)

c) Write $p(\lambda) = (\lambda - \lambda_0)^m q(\lambda)$ to show that $p^{(j)}(\lambda_0) = 0$ for $j = 0, 1, \ldots, m-1$. Conclude that $e^{\lambda_0 t}, te^{\lambda_0 t}, \ldots, t^{m-1}e^{\lambda_0 t}$ solve $L[y] = 0$.

d) Show that these m functions are linearly independent on $\mathbb{R}$.

Solutions: **a)** For every $j \geq 0$, we have $\frac{d^j}{dt^j}e^{\lambda t} = \lambda^j e^{\lambda t}$. With $a_n = 1$,

$$L[e^{\lambda t}] = \sum_{j=0}^{n} a_j \lambda^j e^{\lambda t} = p(\lambda)e^{\lambda t}.$$

b) For fixed t, we have $\frac{\partial^k}{\partial \lambda^k}e^{\lambda t} = t^k e^{\lambda t}$. The operator L differentiates in t and has constant coefficients, so interchanging the derivatives gives

$$\frac{\partial^k}{\partial \lambda^k} L[e^{\lambda t}] = L\left[\frac{\partial^k}{\partial \lambda^k} e^{\lambda t}\right] = L[t^k e^{\lambda t}].$$

On the other hand, the Leibniz rule for the kth derivative of the product $p(\lambda)e^{\lambda t}$ with respect to λ gives

$$\frac{\partial^k}{\partial \lambda^k}\left(p(\lambda)e^{\lambda t}\right) = \sum_{j=0}^{k} \binom{k}{j} p^{(j)}(\lambda) \frac{\partial^{k-j}}{\partial \lambda^{k-j}} e^{\lambda t} = \sum_{j=0}^{k} \binom{k}{j} p^{(j)}(\lambda) t^{k-j} e^{\lambda t}.$$

By a), the two left-hand sides are equal, which proves the identity.

c) Let $0 \le j \le m-1$. The Leibniz rule gives

$$p^{(j)}(\lambda) = \sum_{i=0}^{j} \binom{j}{i} \frac{m!}{(m-i)!} (\lambda - \lambda_0)^{m-i} q^{(j-i)}(\lambda).$$

Every term contains the factor $(\lambda - \lambda_0)^{m-i}$ with $m - i \ge m - j \ge 1$, so $p^{(j)}(\lambda_0) = 0$. Now let $0 \le k \le m-1$. Every index j in the sum in b) satisfies $j \le k \le m-1$, so every term vanishes at $\lambda = \lambda_0$, and $L[t^k e^{\lambda_0 t}] = 0$.

d) Suppose that $\sum_{k=0}^{m-1} \alpha_k t^k e^{\lambda_0 t} = 0$ for every $t \in \mathbb{R}$. Multiplying by $e^{-\lambda_0 t} > 0$ shows that the polynomial $P(t) = \sum_{k=0}^{m-1} \alpha_k t^k$ vanishes at every $t \in \mathbb{R}$. A nonzero polynomial of degree at most $m-1$ has at most $m-1$ roots, so P is the zero polynomial, that is, $\alpha_0 = \cdots = \alpha_{m-1} = 0$. By Lemma 3.1 of the notes, the functions $t^k e^{\lambda_0 t}$, $k = 0, \ldots, m-1$, are linearly independent on $\mathbb{R}$.

Problem 3

Use the method of undetermined coefficients in each part.

a) Find the general solution of $y''(t) - 4y(t) = te^{-t}$.

b) Find the general solution of $y''(t) + 4y(t) = t^2 + 3e^t + 3\sin(2t)$. **Hint:** Find a particular solution for each of the three terms on the right-hand side separately.

c) Solve the initial value problem $y''(t) - 2y'(t) + y(t) = e^t$, with $y(0) = 1$ and $y'(0) = 0$.

Solutions: **a)** The characteristic polynomial $p(\lambda) = \lambda^2 - 4$ has the roots ± 2, so the homogeneous solutions are e^{2t} and e^{-2t}. The number -1 is not a root of p, so we try $y_p(t) = (A + Bt)e^{-t}$. Then

$$y_p'(t) = (B - A - Bt)e^{-t}, \qquad y_p''(t) = (A - 2B + Bt)e^{-t},$$

and

$$y_p''(t) - 4y_p(t) = (-3A - 2B - 3Bt)e^{-t}.$$

This expression equals te^{-t} for every t if and only if $-3B = 1$ and $-3A - 2B = 0$, that is, $B = -\frac{1}{3}$ and $A = \frac{2}{9}$. The general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} + \left(\frac{2}{9} - \frac{t}{3}\right) e^{-t}.$$

b) The characteristic polynomial $p(\lambda) = \lambda^2 + 4$ has the roots $\pm 2i$, so the homogeneous solutions are $\cos(2t)$ and $\sin(2t)$. Write $L[y] = y'' + 4y$. If $L[y_1] = f_1$, $L[y_2] = f_2$, and $L[y_3] = f_3$, then $L[y_1 + y_2 + y_3] = f_1 + f_2 + f_3$ because L is linear. We treat the three terms separately.

- For $f_1(t) = t^2$, the number 0 is not a root of p, so we try $y_1(t) = A_0 + A_1 t + A_2 t^2$. Then

  $$L[y_1](t) = (2A_2 + 4A_0) + 4A_1 t + 4A_2 t^2,$$

  and comparing coefficients with t^2 gives $A_2 = \frac{1}{4}$, $A_1 = 0$, and $A_0 = -\frac{1}{8}$. Hence $y_1(t) = \frac{t^2}{4} - \frac{1}{8}$.

- For $f_2(t) = 3e^t$, the number 1 is not a root of p, so we try $y_2(t) = Ce^t$. Then $L[y_2](t) = 5Ce^t$, so $C = \frac{3}{5}$.

- For $f_3(t) = 3\sin(2t)$, the number $2i$ is a simple root of p, so we try

  $$y_3(t) = t\left(A\cos(2t) + B\sin(2t)\right).$$

  As in Section 3.5 of the notes (with $\omega_0 = 2$),

  $$L[y_3](t) = 2\left(-2A\sin(2t) + 2B\cos(2t)\right).$$

  This expression equals $3\sin(2t)$ if and only if $A = -\frac{3}{4}$ and $B = 0$, so $y_3(t) = -\frac{3}{4}t\cos(2t)$.

The general solution is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t - \frac{3}{4}t\cos(2t).$$

c) The characteristic polynomial $p(\lambda) = (\lambda - 1)^2$ has the double root 1, so the homogeneous solutions are e^t and te^t, and the number 1 has multiplicity $m = 2$. We try $y_p(t) = Ct^2e^t$. Then

$$y_p'(t) = C(t^2 + 2t)e^t, \qquad y_p''(t) = C(t^2 + 4t + 2)e^t,$$

so

$$y_p'' - 2y_p' + y_p = Ce^t\left((t^2 + 4t + 2) - 2(t^2 + 2t) + t^2\right) = 2Ce^t.$$

Hence $C = \frac{1}{2}$, and the general solution is $y(t) = (c_1 + c_2 t)e^t + \frac{1}{2}t^2e^t$. Its derivative is $y'(t) = (c_1 + c_2 + c_2 t)e^t + \left(t + \frac{t^2}{2}\right)e^t$. The initial conditions give $y(0) = c_1 = 1$ and $y'(0) = c_1 + c_2 = 0$, so $c_2 = -1$ and

$$y(t) = \left(1 - t + \frac{t^2}{2}\right)e^t.$$

Problem 4

Use variation of constants in a) and b).

a) Find the general solution of

$$ty''(t) - (t+1)y'(t) + y(t) = t^2, \qquad t > 0.$$

Hint: In Problem 7 of Assignment 3, you showed that $\{1+t, e^t\}$ is a fundamental set of solutions of the homogeneous equation on $(0, \infty)$.

b) Find the general solution of

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2}, \qquad t \in \mathbb{R}.$$

c) Explain why the method of undetermined coefficients does not apply to the equation in b).

Solutions: **a)** The variation-of-constants formulas of Section 3.3.2 of the notes hold for the normalized equation $y'' + a_1(t)y' + a_0(t)y = f(t)$. Dividing by $t > 0$, we obtain

$$y''(t) - \frac{t+1}{t}y'(t) + \frac{1}{t}y(t) = t,$$

so $f(t) = t$. Let $y_1(t) = 1+t$ and $y_2(t) = e^t$. Their Wronskian is $W(t) = (1+t)e^t - e^t = te^t$, and Cramer's rule gives

$$c_1'(t) = -\frac{y_2(t)f(t)}{W(t)} = -\frac{e^t t}{te^t} = -1, \qquad c_2'(t) = \frac{y_1(t)f(t)}{W(t)} = \frac{(1+t)t}{te^t} = (1+t)e^{-t}.$$

Integrating, $c_1(t) = -t + K_1$ and $c_2(t) = -(2+t)e^{-t} + K_2$ with constants $K_1, K_2 \in \mathbb{R}$. (Indeed, $\frac{d}{dt}\left(-(2+t)e^{-t}\right) = -e^{-t} + (2+t)e^{-t} = (1+t)e^{-t}$.) Therefore,

$$y(t) = (K_1 - t)(1+t) + \left(K_2 - (2+t)e^{-t}\right)e^t = K_1(1+t) + K_2e^t - t^2 - 2t - 2.$$

Since $-2t - 2 = -2(1+t)$ is a homogeneous solution, we can absorb it into the first term. The general solution is

$$y(t) = c_1(1+t) + c_2e^t - t^2, \qquad t > 0.$$

As a check, $y_p(t) = -t^2$ gives $t(-2) - (t+1)(-2t) - t^2 = t^2$.

b) The characteristic polynomial $(\lambda - 1)^2$ has the double root 1, so $y_1(t) = e^t$ and $y_2(t) = te^t$, with Wronskian

$$W(t) = e^t(1+t)e^t - e^t \cdot te^t = e^{2t}.$$

With $f(t) = e^t/(1+t^2)$, Cramer's rule gives

$$c_1'(t) = -\frac{te^t f(t)}{e^{2t}} = -\frac{t}{1+t^2}, \qquad c_2'(t) = \frac{e^t f(t)}{e^{2t}} = \frac{1}{1+t^2},$$

so $c_1(t) = -\frac{1}{2}\ln(1+t^2) + K_1$ and $c_2(t) = \arctan t + K_2$. The general solution is

$$y(t) = K_1 e^t + K_2 t e^t - \frac{1}{2} e^t \ln(1+t^2) + t e^t \arctan t.$$

c) The method of undetermined coefficients applies when the inhomogeneity is a polynomial, an exponential, a sine or cosine, or a product of these, that is, a function $P(t)e^{rt}\cos(\omega t)$ or $P(t)e^{rt}\sin(\omega t)$ with a polynomial P and real numbers r and ω; the trial solution then has the same form. The function $f(t) = e^t/(1+t^2)$ is not of this form. Indeed, $f > 0$ on $\mathbb{R}$, whereas a nonzero function of this form with $\omega \neq 0$ vanishes at every zero of $\cos(\omega t)$ or $\sin(\omega t)$. If $\omega = 0$, then $f(t) = P(t)e^{rt}$ would give $e^{(1-r)t} = P(t)(1+t^2)$ for every $t \in \mathbb{R}$. For $r = 1$, this is impossible because a nonconstant polynomial is unbounded and a constant polynomial P makes $P(t)(1+t^2)$ either zero or unbounded, while the left-hand side equals 1. For $r \neq 1$, the left-hand side tends to 0 as $t \to \infty$ or as $t \to -\infty$, so the polynomial $P(t)(1+t^2)$ would be bounded on a half-line, hence constant, hence zero, which contradicts $e^{(1-r)t} > 0$.

The natural trial form $y = P(t)e^t$ also fails directly. For $y = ue^t$ we have $y' = (u+u')e^t$ and $y'' = (u+2u'+u'')e^t$, so

$$y'' - 2y' + y = u'' e^t.$$

The ODE therefore requires $u'' = 1/(1+t^2)$, which no polynomial u satisfies, since u'' would be a polynomial.

Problem 5

a) Let $a_1, a_0 \in \mathbb{R}$. Prove that every solution of $y''(t) + a_1 y'(t) + a_0 y(t) = 0$ satisfies $\lim_{t\to\infty} y(t) = 0$ if and only if $a_1 > 0$ and $a_0 > 0$.

Now let $m > 0$, $k > 0$, and $\gamma \geq 0$, and let y solve the damped oscillator equation $my''(t) + \gamma y'(t) + ky(t) = 0$.

b) Show that the energy $E(t) = \frac{1}{2}my'(t)^2 + \frac{1}{2}ky(t)^2$ satisfies $E'(t) = -\gamma y'(t)^2$ for every $t \in \mathbb{R}$. Conclude that E is nonincreasing, and that E is constant when $\gamma = 0$.

c) Suppose that the oscillator is critically damped, $\gamma^2 = 4km$, and let $\omega_0 = \sqrt{k/m}$. Show that a nonzero solution vanishes at most once. Then suppose that $y(0) = y_0 > 0$ and $y'(0) = v_0$, and show that the mass passes through the equilibrium position at some time $t > 0$ if and only if $v_0 < -\omega_0 y_0$.

Solutions: **a)** Let λ_1 and λ_2 be the roots of $p(\lambda) = \lambda^2 + a_1\lambda + a_0$, counted with multiplicity. We prove two equivalences.

Step 1: Every solution tends to 0 *if and only if* $\operatorname{Re}\lambda_1 < 0$ *and* $\operatorname{Re}\lambda_2 < 0$. Suppose that both roots have negative real part. In each of the three cases of Section 3.2 of the notes, the two functions of the fundamental set tend to 0: $e^{\lambda_j t} \to 0$ for real $\lambda_j < 0$; $|e^{\alpha t}\cos(\beta t)| \leq e^{\alpha t} \to 0$ and $|e^{\alpha t}\sin(\beta t)| \leq e^{\alpha t} \to 0$ for complex roots $\alpha \pm i\beta$ with $\alpha < 0$; and $e^{\lambda_0 t} \to 0$ and $te^{\lambda_0 t} \to 0$ for a double root $\lambda_0 < 0$. Every solution is a linear combination of the two functions, so every solution tends to 0. Conversely, suppose that some root has nonnegative

real part. If this root λ is real, then $y(t) = e^{\lambda t} \geq 1$ for $t \geq 0$ is a solution that does not tend to 0. If the roots are $\alpha \pm i\beta$ with $\beta \neq 0$ and $\alpha \geq 0$, then $y(t) = e^{\alpha t}\cos(\beta t)$ is a solution with $y(2\pi n/|\beta|) = e^{2\pi n\alpha/|\beta|} \geq 1$ for every $n \in \mathbb{N}$, so y does not tend to 0.

Step 2: $\operatorname{Re}\lambda_1 < 0$ *and* $\operatorname{Re}\lambda_2 < 0$ *if and only if* $a_1 > 0$ *and* $a_0 > 0$. Since $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$, we have $a_1 = -(\lambda_1 + \lambda_2)$ and $a_0 = \lambda_1\lambda_2$. If the roots are real, they are both negative if and only if their product is positive (so they have the same sign) and their sum is negative, that is, $a_0 > 0$ and $a_1 > 0$. If the roots are $\alpha \pm i\beta$ with $\beta \neq 0$, then $a_1 = -2\alpha$ and $a_0 = \alpha^2 + \beta^2 > 0$, so $\alpha < 0$ if and only if $a_1 > 0$, and $a_0 > 0$ holds automatically.

Combining the two steps proves the claim.

b) The coefficients are constant, so every solution is defined on $\mathbb{R}$ by Theorem 3.1 of the notes. By the chain rule and the ODE $my'' = -\gamma y' - ky$,

$$E'(t) = my'(t)y''(t) + ky(t)y'(t) = y'(t)\left(my''(t) + ky(t)\right) = y'(t)\left(-\gamma y'(t)\right) = -\gamma y'(t)^2.$$

Since $\gamma \geq 0$, we have $E' \leq 0$ on $\mathbb{R}$, and the mean value theorem shows that E is nonincreasing. If $\gamma = 0$, then $E' \equiv 0$, and E is constant: the undamped oscillator conserves the sum of its kinetic energy $\frac{1}{2}m(y')^2$ and its potential energy $\frac{1}{2}ky^2$.

c) If $\gamma^2 = 4km$, then $\frac{\gamma}{2m} = \frac{2\sqrt{km}}{2m} = \omega_0$, so the characteristic polynomial $m\lambda^2 + \gamma\lambda + k = m(\lambda + \omega_0)^2$ has the double root $-\omega_0$, and every solution has the form $y(t) = (c_1 + c_2 t)e^{-\omega_0 t}$. Let y be nonzero, so that $(c_1, c_2) \neq (0, 0)$. The exponential factor never vanishes, so $y(t) = 0$ if and only if $c_1 + c_2 t = 0$. If $c_2 = 0$, then $c_1 \neq 0$ and y never vanishes; if $c_2 \neq 0$, then y vanishes only at $t_* = -c_1/c_2$. In both cases, y vanishes at most once.

Now $y(0) = c_1 = y_0$ and $y'(0) = c_2 - \omega_0 c_1 = v_0$, so $c_2 = v_0 + \omega_0 y_0$. By the previous paragraph, y vanishes at some $t > 0$ if and only if $c_2 \neq 0$ and $-y_0/c_2 > 0$. Since $y_0 > 0$, this holds if and only if $c_2 < 0$, that is, $v_0 < -\omega_0 y_0$.

Problem 6

Consider the forced oscillator

$$y''(t) + 9y(t) = 5\cos(\omega t), \qquad y(0) = 0, \quad y'(0) = 0.$$

a) Let $\omega = 2$. Solve the initial value problem, and write the solution as a product of two sines.

b) Let $\omega = 3$. Solve the initial value problem, and describe the behavior of the solution as $t \to \infty$.

c) Now add damping: consider $y''(t) + y'(t) + 9y(t) = 5\cos(\omega t)$. Find the amplitude $R(\omega)$ of the steady-state solution, the frequency $\omega_{\max}$ at which R is largest, and the value $R(\omega_{\max})$.

Solutions: The characteristic polynomial of the homogeneous equation is $p(\lambda) = \lambda^2 + 9$, with roots $\pm 3i$, so the homogeneous solutions are $\cos(3t)$ and $\sin(3t)$.

a) The number $2i$ is not a root of p, so we try $y_p(t) = A\cos(2t) + B\sin(2t)$. Substitution gives $(9-4)(A\cos(2t) + B\sin(2t)) = 5\cos(2t)$, so $A = 1$ and $B = 0$. The general solution is $y(t) = \cos(2t) + c_1\cos(3t) + c_2\sin(3t)$. The initial conditions give $y(0) = 1 + c_1 = 0$ and $y'(0) = 3c_2 = 0$, so

$$y(t) = \cos(2t) - \cos(3t) = 2\sin\left(\frac{t}{2}\right)\sin\left(\frac{5t}{2}\right),$$

where we used $\cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2}$ with $a = 2t$ and $b = 3t$. The solution oscillates with frequency $\frac{5}{2}$ inside the slowly varying envelope $\pm 2\sin(t/2)$: this is the phenomenon of beats.

b) The number $3i$ is a simple root of p, so we try $y_p(t) = t\,(A\cos(3t) + B\sin(3t))$. As in Section 3.5 of the notes (with $\omega_0 = 3$), $y_p'' + 9y_p = 2\,(-3A\sin(3t) + 3B\cos(3t))$, which equals $5\cos(3t)$ if and only if $A = 0$ and $B = \frac{5}{6}$. The general solution is $y(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{5}{6}t\sin(3t)$. The initial conditions give $y(0) = c_1 = 0$ and $y'(0) = 3c_2 = 0$, so

$$y(t) = \frac{5}{6}t\sin(3t).$$

The solution oscillates with an amplitude that grows linearly in t. It vanishes at $t = n\pi/3$ for every $n \in \mathbb{N}$, while $y(t_n) = \frac{5}{6}t_n$ at $t_n = \frac{\pi/2 + 2\pi n}{3}$. Hence $y(t)$ has no limit as $t \to \infty$, and $\limsup_{t\to\infty} |y(t)| = \infty$: the oscillator is in resonance.

c) The roots of $\lambda^2 + \lambda + 9$ are $-\frac{1}{2} \pm i\frac{\sqrt{35}}{2}$, so every homogeneous solution tends to 0, and $i\omega$ is not a root. We try $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Comparing the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ after substitution gives

$$(9-\omega^2)A + \omega B = 5, \qquad -\omega A + (9-\omega^2)B = 0,$$

so

$$A = \frac{5(9-\omega^2)}{\Delta}, \qquad B = \frac{5\omega}{\Delta}, \qquad \Delta = (9-\omega^2)^2 + \omega^2 > 0.$$

The steady state is $y_p(t) = R\cos(\omega t - \delta)$ with amplitude

$$R(\omega) = \sqrt{A^2 + B^2} = \frac{5}{\sqrt{(9-\omega^2)^2 + \omega^2}}.$$

The amplitude is largest when $g(u) = (9-u)^2 + u$ is smallest, where $u = \omega^2 > 0$. The function g is a convex quadratic with $g'(u) = 2u - 17$, so its minimum is at $u = \frac{17}{2}$, with $g(\frac{17}{2}) = \frac{1}{4} + \frac{17}{2} = \frac{35}{4}$. Therefore,

$$\omega_{\max} = \sqrt{\frac{17}{2}}, \qquad R(\omega_{\max}) = \frac{5}{\sqrt{35}/2} = \frac{10}{\sqrt{35}}.$$

This agrees with Section 3.5 of the notes: with $m = 1$, $\gamma = 1$, $\omega_0 = 3$, and $F_0 = 5$, we have $\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \frac{17}{2}$ and $\omega_d = \sqrt{9 - \frac{1}{4}} = \frac{\sqrt{35}}{2}$, so $R(\omega_{\max}) = \frac{F_0}{\gamma\omega_d} = \frac{10}{\sqrt{35}}$.

Problem 7

Let $\omega_0 > 0$ and $T > 0$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic, that is, $f(t+T) = f(t)$ for every $t \in \mathbb{R}$. Consider the equation

$$y''(t) + \omega_0^2 y(t) = f(t), \qquad t \in \mathbb{R}.$$

a) Let y be a solution. Show that y is T-periodic if and only if $y(T) = y(0)$ and $y'(T) = y'(0)$. **Hint:** Show that $z(t) = y(t+T)$ is also a solution, and use Theorem 3.1 of the notes.

b) Show that the variation-of-constants formula of Section 3.3.2 of the notes, with the fundamental set $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$ and $t_0 = 0$, gives the particular solution

$$y_p(t) = \frac{1}{\omega_0} \int_0^t \sin\left(\omega_0(t-\tau)\right) f(\tau)\, d\tau,$$

and that $y_p(0) = y_p'(0) = 0$. **Hint:** $\frac{d}{dt}\int_0^t K(t,\tau)\,d\tau = K(t,t) + \int_0^t \frac{\partial K}{\partial t}(t,\tau)\,d\tau$ when K and $\frac{\partial K}{\partial t}$ are continuous.

c) Write every solution as $y = y_p + c_1\cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. Use a) to show that y is T-periodic if and only if (c_1, c_2) solves a linear system of two equations whose determinant equals $2\omega_0\left(1 - \cos(\omega_0 T)\right)$. Conclude that if $\omega_0 T$ is not an integer multiple of 2π, then the equation has exactly one T-periodic solution.

d) Let $f(t) = \cos t$ and $T = 2\pi$. Find all 2π-periodic solutions in each of the cases $\omega_0 = \sqrt{2}$, $\omega_0 = 2$, and $\omega_0 = 1$.

Solutions: **a)** If y is T-periodic, then $y(T) = y(0)$, and differentiating $y(t+T) = y(t)$ gives $y'(T) = y'(0)$. Conversely, suppose that $y(T) = y(0)$ and $y'(T) = y'(0)$, and let $z(t) = y(t+T)$. For every $t \in \mathbb{R}$,

$$z''(t) + \omega_0^2 z(t) = y''(t+T) + \omega_0^2 y(t+T) = f(t+T) = f(t),$$

so z solves the same ODE. Moreover, $z(0) = y(T) = y(0)$ and $z'(0) = y'(T) = y'(0)$. The functions y and z therefore solve the same initial value problem on $I = \mathbb{R}$, and the uniqueness part of Theorem 3.1 of the notes gives $z = y$, that is, $y(t+T) = y(t)$ for every $t \in \mathbb{R}$.

b) With $y_1(t) = \cos(\omega_0 t)$ and $y_2(t) = \sin(\omega_0 t)$, the Wronskian is $W = \omega_0\cos^2(\omega_0 t) + \omega_0 \sin^2(\omega_0 t) = \omega_0$, and

$$y_1(\tau)y_2(t) - y_1(t)y_2(\tau) = \sin(\omega_0 t)\cos(\omega_0\tau) - \cos(\omega_0 t)\sin(\omega_0\tau) = \sin\left(\omega_0(t-\tau)\right).$$

The boxed formula of Section 3.3.2 with $t_0 = 0$ therefore gives the stated y_p. Clearly, $y_p(0) = 0$. The function $K(t,\tau) = \frac{1}{\omega_0}\sin(\omega_0(t-\tau))f(\tau)$ and its partial derivative $\frac{\partial K}{\partial t}(t,\tau) = \cos(\omega_0(t-\tau))f(\tau)$ are continuous, and $K(t,t) = 0$, so the hint gives

$$y_p'(t) = \int_0^t \cos\left(\omega_0(t-\tau)\right) f(\tau)\, d\tau,$$

and $y_p'(0) = 0$.

c) By Theorem 3.3 of the notes, every solution has the form $y = y_p + c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$, and the constants are unique because $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ are linearly independent. Let $\theta = \omega_0 T$. Using $y_p(0) = y_p'(0) = 0$, we find

$$\begin{aligned} y(T) - y(0) &= y_p(T) + c_1(\cos\theta - 1) + c_2\sin\theta, \\ y'(T) - y'(0) &= y_p'(T) - c_1\omega_0\sin\theta + c_2\omega_0(\cos\theta - 1). \end{aligned}$$

By a), y is T-periodic if and only if

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\omega_0\sin\theta & \omega_0(\cos\theta - 1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} y_p(T) \\ y_p'(T) \end{pmatrix}.$$

The determinant of this matrix is

$$\omega_0(\cos\theta - 1)^2 + \omega_0\sin^2\theta = \omega_0\left(\cos^2\theta + \sin^2\theta - 2\cos\theta + 1\right) = 2\omega_0(1 - \cos\theta).$$

If $\theta = \omega_0 T$ is not an integer multiple of 2π, then $\cos\theta \neq 1$, the determinant is nonzero, and the system has exactly one solution (c_1, c_2). Hence the ODE has exactly one T-periodic solution.

d) Let $f(t) = \cos t$ and $T = 2\pi$.

- Case $\omega_0 = \sqrt{2}$. If $2\pi\sqrt{2} = 2\pi k$ for some integer k, then $\sqrt{2} = k$ would be rational, so $\omega_0 T$ is not an integer multiple of 2π, and by c) there is exactly one 2π-periodic solution. The number i is not a root of $\lambda^2 + 2$, and the trial form $A\cos t$ gives $(2-1)A = 1$. The function $y(t) = \cos t$ is a 2π-periodic solution, hence the only one.

- Case $\omega_0 = 2$. Now $\omega_0 T = 4\pi$, and the determinant in c) vanishes. The trial form $A\cos t$ gives $(4-1)A = 1$, so every solution has the form

  $$y(t) = \frac{1}{3}\cos t + c_1\cos(2t) + c_2\sin(2t),$$

  and each of these functions is 2π-periodic. Every solution is 2π-periodic, so there are infinitely many 2π-periodic solutions.

- Case $\omega_0 = 1$. Now $\omega_0 T = 2\pi$, so $\cos\theta = 1$ and $\sin\theta = 0$, and the matrix in c) is zero. Section 3.5 of the notes (with $\omega_0 = 1$ and $F_0/m = 1$) shows that $\frac{1}{2}t\sin t$ solves the ODE with zero initial data at $t = 0$. The function y_p of b) solves the same initial value problem, so the uniqueness part of Theorem 3.1 gives $y_p(t) = \frac{1}{2}t\sin t$, so $y_p'(t) = \frac{1}{2}(\sin t + t\cos t)$ and $y_p'(2\pi) = \pi$. The second equation of the system in c) reads $0 = -\pi$, which is impossible. Hence there is no 2π-periodic solution. This is consistent with the resonance of Section 3.5: every solution has the form $c_1\cos t + c_2\sin t + \frac{1}{2}t\sin t$ and is unbounded.

Problem 8

Prove Theorem 3.2 of the notes in your own words. Let I be an open interval, let a_1 and a_0 be continuous on I, and consider the homogeneous equation

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0, \qquad t \in I.$$

Theorem (Structure of the solution set)**.** *(a) The solutions of the homogeneous equation form a vector space of dimension two.*

(b) If y_1 and y_2 are linearly independent solutions of the homogeneous equation, then every solution y can be written as $y = c_1y_1 + c_2y_2$ for unique constants $c_1, c_2 \in \mathbb{R}$.

You may use Theorem 3.1 (existence and uniqueness), Proposition 3.1 (superposition principle), Lemma 3.1, and Corollary 3.1 of the notes without proof. Justify every step.

Solution: Let S denote the set of solutions of the homogeneous equation on I.

(a) The zero function belongs to S, and by Proposition 3.1, every linear combination of elements of S belongs to S. Hence S is a vector space (a subspace of the space of all functions $I \to \mathbb{R}$). Fix $t_0 \in I$. By the existence part of Theorem 3.1, there exist $z_1, z_2 \in S$ with

$$z_1(t_0) = 1, \quad z_1'(t_0) = 0, \qquad z_2(t_0) = 0, \quad z_2'(t_0) = 1.$$

We show that $\{z_1, z_2\}$ is a basis of S.

Linear independence. Suppose that $\alpha_1 z_1(t) + \alpha_2 z_2(t) = 0$ for every $t \in I$. Evaluating at t_0 gives $\alpha_1 = 0$. Differentiating the identity and evaluating at t_0 gives $\alpha_2 = 0$. By Lemma 3.1, z_1 and z_2 are linearly independent.

Spanning. Let $y \in S$, and let $w = y - y(t_0)z_1 - y'(t_0)z_2$. Then $w \in S$ by Proposition 3.1, and

$$w(t_0) = y(t_0) - y(t_0) \cdot 1 - y'(t_0) \cdot 0 = 0, \qquad w'(t_0) = y'(t_0) - y(t_0) \cdot 0 - y'(t_0) \cdot 1 = 0.$$

The zero function solves the same initial value problem, so the uniqueness part of Theorem 3.1 gives $w \equiv 0$ on I. Thus $y = y(t_0)z_1 + y'(t_0)z_2$ lies in $\mathrm{Span}(\{z_1, z_2\})$.

Therefore, $\{z_1, z_2\}$ is a basis of S, and $\dim S = 2$.

(b) Let $y_1, y_2 \in S$ be linearly independent, and let $y \in S$.

Existence of the constants. By Corollary 3.1, the Wronskian $W = W[y_1, y_2]$ satisfies $W(t_0) \neq 0$. The linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}$$

has determinant $W(t_0) \neq 0$, so it has a solution (c_1, c_2). The function $c_1y_1 + c_2y_2$ belongs to S by Proposition 3.1, and it has the same value and the same derivative at t_0 as y. By the uniqueness part of Theorem 3.1, $y = c_1y_1 + c_2y_2$ on I.

Uniqueness of the constants. Suppose that $y = c_1y_1 + c_2y_2 = d_1y_1 + d_2y_2$ on I. Then $(c_1 - d_1)y_1 + (c_2 - d_2)y_2 = 0$ on I, and Lemma 3.1 gives $c_1 = d_1$ and $c_2 = d_2$.

Problem 9

A function $f\colon [0,\infty) \to \mathbb{R}$ is *piecewise continuous* if, on every bounded interval, f has at most finitely many discontinuities, each with finite one-sided limits. Let f be piecewise continuous, and suppose that there exist constants $K > 0$, $\alpha \in \mathbb{R}$, and $M \geq 0$ such that

$$|f(t)| \leq Ke^{\alpha t} \qquad \forall\, t \geq M.$$

For $s \in \mathbb{R}$, consider the improper integral

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

a) Prove that $F(s)$ is well defined, that is, the improper integral converges, for every $s > \alpha$. You may use without proof that an improper integral $\int_0^\infty g(t)\, dt$ of a piecewise continuous function g converges whenever $\int_0^\infty |g(t)|\, dt$ converges.

b) For each of the following functions, find the values of α for which the bound above holds (for suitable K and M), and compute $F(s)$ for every s at which a) guarantees convergence:

(1) $f(t) = 1$;

(2) $f(t) = t^n$, with $n \in \mathbb{N}$ (**Hint:** integrate by parts and use induction on n);

(3) $f(t) = e^{\lambda t}$, with $\lambda \in \mathbb{C}$ (here $|\cdot|$ denotes the modulus of a complex number, and the result in a) holds for complex-valued f as well);

(4) $f(t) = \cos(\omega t)$ and $f(t) = \sin(\omega t)$, with $\omega \in \mathbb{R}$.

Solutions: **a)** Let $s > \alpha$. We first show that f is bounded on $[0, M]$. Let $t_1 < \cdots < t_N$ be the discontinuities of f in $[0, M]$, and set $t_0 = 0$ and $t_{N+1} = M$. On each interval (t_i, t_{i+1}), the function f is continuous and has finite one-sided limits at the endpoints, so it extends to a continuous function on $[t_i, t_{i+1}]$, which is bounded by the extreme value theorem. Finitely many such bounds, together with the finitely many values $f(t_i)$, give a bound B with $|f(t)| \leq B$ for every $t \in [0, M]$. Hence

$$\int_0^M |e^{-st} f(t)|\, dt \leq B \int_0^M e^{-st}\, dt < \infty.$$

For $R > M$, the bound on $|f|$ gives

$$\int_M^R |e^{-st} f(t)|\, dt \leq K \int_M^R e^{-(s-\alpha)t}\, dt = \frac{K}{s-\alpha}\left(e^{-(s-\alpha)M} - e^{-(s-\alpha)R}\right) \leq \frac{Ke^{-(s-\alpha)M}}{s-\alpha},$$

where we used $s - \alpha > 0$. The left-hand side is nondecreasing in R and bounded, so it converges as $R \to \infty$. Thus $\int_0^\infty |e^{-st} f(t)|\, dt < \infty$, and the improper integral defining $F(s)$ converges by the fact we were allowed to use.

b) Throughout, $R \to \infty$ denotes the limit in the improper integral $\int_0^\infty = \lim_{R\to\infty} \int_0^R$.

(1) The bound $1 \le Ke^{\alpha t}$ holds for all $t \ge M$ (with suitable K and M) if and only if $\alpha \ge 0$, because $e^{\alpha t} \to 0$ when $\alpha < 0$. Taking $\alpha = 0$, the integral converges for every $s > 0$, and

$$F(s) = \lim_{R\to\infty} \int_0^R e^{-st}\,dt = \lim_{R\to\infty} \frac{1 - e^{-sR}}{s} = \frac{1}{s}.$$

(2) Let $n \ge 1$. For $\alpha \le 0$, the bound fails because $t^n \to \infty$ while $Ke^{\alpha t} \le K$. For $\alpha > 0$, the function $t \mapsto t^n e^{-\alpha t}$ is continuous on $[0, \infty)$, increases on $[0, n/\alpha]$ and decreases on $[n/\alpha, \infty)$, so $t^n \le Ke^{\alpha t}$ for every $t \ge 0$ with $K = \left(\frac{n}{\alpha e}\right)^n$. The bound therefore holds exactly for $\alpha > 0$, and the integral converges for every $s > 0$. Fix $s > 0$, and let $F_n(s) = \int_0^\infty e^{-st} t^n\,dt$. Integration by parts gives

$$\int_0^R e^{-st} t^n\,dt = \left[-\frac{t^n e^{-st}}{s}\right]_0^R + \frac{n}{s}\int_0^R e^{-st} t^{n-1}\,dt.$$

The boundary term equals $-R^n e^{-sR}/s \to 0$ as $R \to \infty$, so $F_n(s) = \frac{n}{s} F_{n-1}(s)$. Since $F_0(s) = \frac{1}{s}$ by (1), induction on n gives

$$F_n(s) = \frac{n!}{s^{n+1}}, \qquad s > 0.$$

(3) Write $\lambda = \mu + i\nu$ with $\mu, \nu \in \mathbb{R}$. Then $|e^{\lambda t}| = e^{\mu t}|e^{i\nu t}| = e^{\mu t}$, so the bound holds if and only if $\alpha \ge \operatorname{Re}\lambda$. Let $s > \operatorname{Re}\lambda$. Since $\frac{d}{dt} e^{(\lambda - s)t} = (\lambda - s) e^{(\lambda - s)t}$ (as in Section 3.2 of the notes),

$$\int_0^R e^{-st} e^{\lambda t}\,dt = \frac{e^{(\lambda - s)R} - 1}{\lambda - s},$$

and $|e^{(\lambda - s)R}| = e^{(\operatorname{Re}\lambda - s)R} \to 0$ as $R \to \infty$. Therefore,

$$F(s) = \frac{1}{s - \lambda}, \qquad s > \operatorname{Re}\lambda.$$

(4) Let $\omega \ne 0$. Since $|\cos(\omega t)| \le 1$ and $|\sin(\omega t)| \le 1$, the bound holds for every $\alpha \ge 0$. It fails for $\alpha < 0$, because $|\cos(\omega t)| = 1$ at $t = 2\pi k/|\omega|$ and $|\sin(\omega t)| = 1$ at $t = (\pi/2 + 2\pi k)/|\omega|$ for every $k \in \mathbb{N}$, while $Ke^{\alpha t} \to 0$. (If $\omega = 0$, then $\cos(\omega t) = 1$ is covered by (1), and $\sin(\omega t) = 0$ satisfies the bound for every α, with $F \equiv 0$.) Let $s > 0$. By Euler's formula, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, and the integral of a complex-valued function is the integral of its real part plus i times the integral of its imaginary part. Applying (3) with $\lambda = i\omega$,

$$\int_0^\infty e^{-st}\cos(\omega t)\,dt + i\int_0^\infty e^{-st}\sin(\omega t)\,dt = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2}.$$

Comparing real and imaginary parts,

$$\int_0^\infty e^{-st}\cos(\omega t)\,dt = \frac{s}{s^2 + \omega^2}, \qquad \int_0^\infty e^{-st}\sin(\omega t)\,dt = \frac{\omega}{s^2 + \omega^2}, \qquad s > 0.$$