

MATH 441: Practice Quiz 1 — Solutions

Problem 1 (12 points)

Consider the initial value problem

$$y'(t) = -4y(t), \quad y(0) = 2.$$

- a) Solve the initial value problem. Then apply Euler's method with step size $h > 0$, and show that $y_n = 2(1 - 4h)^n$ for every $n \geq 0$.
- b) For which step sizes $h > 0$ does $y_n \rightarrow 0$ as $n \rightarrow \infty$? For which step sizes $h > 0$ is $y_n > 0$ for every $n \geq 0$? Describe the behavior of the sequence $\{y_n\}_{n=0}^{\infty}$ for $h = 3/4$, and compare it with the behavior of the exact solution.
- c) The *implicit* Euler method reads $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$. Apply it to this problem, show that $y_n = 2(1 + 4h)^{-n}$ for every $n \geq 0$, and conclude that $y_n \rightarrow 0$ as $n \rightarrow \infty$ for every step size $h > 0$.

Solutions: **a)** The ODE is linear and homogeneous with constant coefficient -4 , so $y(t) = Le^{-4t}$, and the initial condition gives $L = 2$. Hence $y(t) = 2e^{-4t}$.

Here $f(t, y) = -4y$, so Euler's method reads

$$y_{n+1} = y_n + h(-4y_n) = (1 - 4h)y_n, \quad n = 0, 1, 2, \dots$$

Since $y_0 = 2$, induction on n gives $y_n = 2(1 - 4h)^n$ for every $n \geq 0$.

b) A geometric sequence q^n tends to 0 if and only if $|q| < 1$. With $q = 1 - 4h$ and $h > 0$, the condition $|1 - 4h| < 1$ reads $-1 < 1 - 4h$, that is,

$$y_n \rightarrow 0 \iff 0 < h < \frac{1}{2}.$$

Every y_n is positive if and only if $q = 1 - 4h > 0$, that is, if and only if $0 < h < \frac{1}{4}$. (For $h = 1/4$, we get $y_n = 0$ for every $n \geq 1$, and for $h > 1/4$, the sign of y_n alternates.)

For $h = 3/4$, we get $q = -2$ and $y_n = 2(-2)^n$. The sequence alternates in sign and $|y_n| = 2^{n+1} \rightarrow \infty$, whereas the exact solution $2e^{-4t}$ is positive and decreases to 0. The approximation is therefore qualitatively wrong for this step size.

c) The implicit Euler method reads $y_{n+1} = y_n - 4hy_{n+1}$, that is, $(1 + 4h)y_{n+1} = y_n$, or

$$y_{n+1} = \frac{y_n}{1 + 4h}, \quad n = 0, 1, 2, \dots$$

Since $y_0 = 2$, induction on n gives $y_n = 2(1 + 4h)^{-n}$. For every $h > 0$, we have $0 < (1 + 4h)^{-1} < 1$, so $y_n \rightarrow 0$ as $n \rightarrow \infty$. (Moreover, every y_n is positive, as is the exact solution.)

Problem 2 (13 points)

Consider the autonomous ODE

$$y'(t) = 4y(t) - y(t)^3.$$

- a) Find all equilibria, draw the phase line, and classify each equilibrium as stable or unstable.
- b) The ODE is a Bernoulli equation. Let $y_0 > 0$, and let y be the solution with $y(0) = y_0$. Use the substitution $v(t) = y(t)^{-2}$ to show that

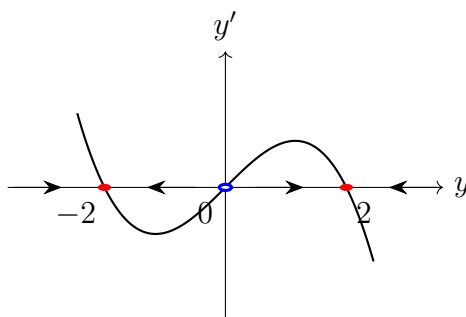
$$y(t) = \frac{1}{\sqrt{\frac{1}{4} + (y_0^{-2} - \frac{1}{4})e^{-8t}}}, \quad t \geq 0.$$

- c) Compute $\lim_{t \rightarrow \infty} y(t)$ from the formula in b), and check that your answer agrees with the phase line in a).

Solutions: a) Write $f(y) = 4y - y^3 = y(2 - y)(2 + y)$. The equilibria are the roots of f , namely $y = -2$, $y = 0$ and $y = 2$. The sign of f on each of the four intervals between consecutive roots follows from the signs of the three factors:

	$y < -2$	$-2 < y < 0$	$0 < y < 2$	$y > 2$
y	−	−	+	+
$2 - y$	+	+	+	−
$2 + y$	−	+	+	+
y'	+	−	+	−

Hence the flow points to the right for $y < -2$ and for $0 < y < 2$, and to the left for $-2 < y < 0$ and for $y > 2$.



(The vertical axis is scaled by a factor $1/2$.) The equilibria $y = -2$ and $y = 2$ are stable: the flow points toward each of them from both sides. The equilibrium $y = 0$ is unstable: the flow points away from it on both sides.

b) Let $[0, t_1)$, with $0 < t_1 \leq \infty$, be the largest interval starting at 0 on which y is defined and positive; it is nonempty because $y(0) = y_0 > 0$ and y is continuous. On $[0, t_1)$, the function $v = y^{-2}$ is well defined and continuously differentiable, and the chain rule and the ODE give

$$v'(t) = -2y(t)^{-3}y'(t) = -2y(t)^{-3}(4y(t) - y(t)^3) = -8y(t)^{-2} + 2 = -8v(t) + 2.$$

This is a linear ODE for v , with $v(0) = y_0^{-2}$. The constant function $v \equiv \frac{1}{4}$ solves it, and the homogeneous solutions are Le^{-8t} , so every solution has the form $v(t) = \frac{1}{4} + Le^{-8t}$, and the initial condition gives $L = y_0^{-2} - \frac{1}{4}$:

$$v(t) = \frac{1}{4} + \left(y_0^{-2} - \frac{1}{4}\right)e^{-8t}, \quad 0 \leq t < t_1.$$

The right-hand side is positive for every $t \geq 0$: if $y_0 \leq 2$, then it is at least $\frac{1}{4}$; if $y_0 > 2$, then $y_0^{-2} - \frac{1}{4} < 0$ and $e^{-8t} \leq 1$, so it is at least $\frac{1}{4} + (y_0^{-2} - \frac{1}{4}) = y_0^{-2} > 0$. Since $y > 0$ on $[0, t_1)$, we obtain $y = v^{-1/2}$ there, which is the stated formula. Finally, y cannot vanish: if $t_1 < \infty$ and $y(t_1) = 0$, then continuity would give $y(t_1) = \lim_{t \rightarrow t_1^-} v(t)^{-1/2} = \left(\frac{1}{4} + (y_0^{-2} - \frac{1}{4})e^{-8t_1}\right)^{-1/2} > 0$, a contradiction. Hence the formula holds for every $t \geq 0$ at which y is defined. Conversely, the formula defines a positive, continuously differentiable function on $[0, \infty)$, and reading the computation above backward shows that it solves the ODE with $y(0) = y_0$; so the solution is defined for every $t \geq 0$.

c) As $t \rightarrow \infty$, $e^{-8t} \rightarrow 0$, so $v(t) \rightarrow \frac{1}{4}$ and $y(t) \rightarrow 2$. This agrees with the phase line: if $0 < y_0 < 2$, the flow points to the right and the solution increases to the stable equilibrium 2; if $y_0 > 2$, the flow points to the left and the solution decreases to 2; and if $y_0 = 2$, the solution is the constant 2.

Problem 3 (12 points)

Theorem 2.2 (Picard–Lindelöf) and the definition of the Picard iterates are stated on the Notes sheet at the end of the quiz.

- a) Consider the initial value problem $y'(t) = 2\sqrt{|y(t)|}$, $y(0) = 0$. For every $c \geq 0$, show that the function

$$y_c(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq c, \\ (t - c)^2 & \text{if } t > c, \end{cases}$$

is a continuously differentiable solution on $[0, \infty)$. Conclude that the initial value problem has infinitely many solutions.

- b) Which of the assumptions (A1), (A2) and (A3) of Theorem 2.2 fail for the problem in a) on the rectangle $D_T = [0, T] \times [-C, C]$? Explain why a) does not contradict Theorem 2.2.

- c) Now consider the initial value problem $y'(t) = 3t^2(1 + y(t))$, $y(0) = 0$. Compute the Picard iterates φ_1 , φ_2 and φ_3 . Guess the limit of φ_k as $k \rightarrow \infty$, and check that it solves the initial value problem.

Solutions: **a)** Fix $c \geq 0$. For $0 \leq t < c$, we have $y_c(t) = 0$, so $y'_c(t) = 0 = 2\sqrt{|y_c(t)|}$. For $t > c$, we have $y'_c(t) = 2(t - c) = 2\sqrt{(t - c)^2} = 2\sqrt{|y_c(t)|}$, because $t - c > 0$. At $t = c$ (with one-sided derivative if $c = 0$), the difference quotients satisfy

$$\frac{y_c(c + s) - y_c(c)}{s} = \begin{cases} 0 & \text{if } s < 0, \\ s & \text{if } s > 0, \end{cases}$$

so $y'_c(c) = 0 = 2\sqrt{|y_c(c)|}$. The derivative $y'_c(t)$ equals 0 for $t \leq c$ and $2(t - c)$ for $t > c$, which is continuous on $[0, \infty)$. Finally, $y_c(0) = 0$. Hence each y_c is a continuously differentiable solution on $[0, \infty)$. Different values of c give different functions (for $c_1 < c_2$, the function y_{c_1} is positive on $(c_1, c_2]$ while y_{c_2} vanishes there), so the initial value problem has infinitely many solutions.

b) Here $f(t, y) = 2\sqrt{|y|}$, $t_0 = 0$ and $y_0 = 0$. Assumption (A1) holds, because $y \mapsto \sqrt{|y|}$ is continuous on $\mathbb{R}$. Assumption (A2) holds, because $f(t, 0) = 0$ for every t . Assumption (A3) fails for every $C > 0$: if $|f(t, y) - f(t, 0)| \leq L|y - 0|$ held for every $y \in [-C, C]$, then for $0 < y \leq C$ we would get $2\sqrt{y} \leq Ly$, that is, $2 \leq L\sqrt{y}$, which fails for $y < 4/L^2$. Theorem 2.2 gives sufficient conditions for uniqueness. Its assumptions are not satisfied here, so the theorem says nothing about this problem, and a) does not contradict it. (Peano's theorem, which only requires continuity, still guarantees existence.)

c) Here $f(t, y) = 3t^2(1 + y)$, $t_0 = 0$, $y_0 = 0$ and $\varphi_0(t) = 0$. Then

$$\begin{aligned} \varphi_1(t) &= \int_0^t 3\tau^2 d\tau = t^3, \\ \varphi_2(t) &= \int_0^t 3\tau^2(1 + \tau^3) d\tau = t^3 + \frac{t^6}{2}, \\ \varphi_3(t) &= \int_0^t 3\tau^2 \left(1 + \tau^3 + \frac{\tau^6}{2}\right) d\tau = t^3 + \frac{t^6}{2} + \frac{t^9}{6}. \end{aligned}$$

The pattern is $\varphi_k(t) = \sum_{j=1}^k \frac{t^{3j}}{j!}$, the partial sums of the Taylor series of $e^{t^3} - 1$. We therefore guess that $\varphi_k(t) \rightarrow y(t) = e^{t^3} - 1$. Indeed, $y(0) = 0$ and

$$y'(t) = 3t^2 e^{t^3} = 3t^2 (1 + y(t)),$$

so y solves the initial value problem.

Problem 4 (13 points)

Consider the ODE

$$(1 + t)y''(t) + ty'(t) - y(t) = 0.$$

- a) Verify that $y_1(t) = t$ and $y_2(t) = e^{-t}$ are solutions for every $t \in \mathbb{R}$.
- b) Compute the Wronskian $W[y_1, y_2](t)$, and show that $\{y_1, y_2\}$ is a fundamental set of solutions on $I = (-1, \infty)$.
- c) Write the ODE in the form $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on $I = (-1, \infty)$, and check that your Wronskian satisfies Abel's identity.
- d) The Wronskian vanishes at $t = -1$ but at no other point. Explain why this does not contradict Corollary 3.1, which is stated on the Notes sheet at the end of the quiz.
- e) Find the solution on $I = (-1, \infty)$ that satisfies $y(0) = 1$ and $y'(0) = 1$.

Solutions: **a)** For $y_1(t) = t$, we have $y_1' = 1$ and $y_1'' = 0$, so $(1+t) \cdot 0 + t \cdot 1 - t = 0$. For $y_2(t) = e^{-t}$, we have $y_2' = -e^{-t}$ and $y_2'' = e^{-t}$, so $(1+t)e^{-t} - te^{-t} - e^{-t} = 0$. Both identities hold for every $t \in \mathbb{R}$.

b) The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -te^{-t} - e^{-t} = -(1+t)e^{-t}.$$

On $I = (-1, \infty)$, dividing the ODE by $1+t \neq 0$ gives an equation with continuous coefficients (see c)), and $W(t) = -(1+t)e^{-t} < 0$ for every $t \in I$. By Corollary 3.1, y_1 and y_2 are linearly independent solutions on I , that is, $\{y_1, y_2\}$ is a fundamental set of solutions on I .

c) Dividing by $1+t$ gives

$$y''(t) + \frac{t}{1+t}y'(t) - \frac{1}{1+t}y(t) = 0, \quad a_1(t) = \frac{t}{1+t}, \quad a_0(t) = -\frac{1}{1+t},$$

and both coefficients are continuous on I . Abel's identity with $t_0 = 0$ predicts $W(t) = W(0)e^{-\int_0^t a_1(\tau) d\tau}$. Since $\frac{\tau}{1+\tau} = 1 - \frac{1}{1+\tau}$,

$$\int_0^t \frac{\tau}{1+\tau} d\tau = t - \ln(1+t), \quad t > -1,$$

so $e^{-\int_0^t a_1(\tau) d\tau} = e^{-t+\ln(1+t)} = (1+t)e^{-t}$. With $W(0) = -1$, Abel's identity gives $W(t) = -(1+t)e^{-t}$, in agreement with b).

d) Corollary 3.1 applies to solutions of $y'' + a_1(t)y' + a_0(t)y = 0$ whose coefficients are continuous on an open interval I . The coefficient of y'' in the original ODE vanishes at $t = -1$, and the normalized coefficients $a_1(t) = t/(1+t)$ and $a_0(t) = -1/(1+t)$ are not defined at $t = -1$. Hence no open interval containing $t = -1$ satisfies the hypotheses of Corollary 3.1, and the corollary says nothing about $t = -1$. On each of the intervals $(-\infty, -1)$ and $(-1, \infty)$, the hypotheses hold, and W indeed vanishes nowhere there.

e) By b), every solution on I has the form $y(t) = c_1t + c_2e^{-t}$. The initial conditions give $y(0) = c_2 = 1$ and $y'(0) = c_1 - c_2 = 1$, so $c_1 = 2$ and

$$y(t) = 2t + e^{-t}.$$