

MATH 441: Practice Quiz 1

Duration: 50 minutes

Please note:

- The practice quiz contains four problems, for a total of 50 points.
- The material covered is Chapter 2 and Section 3.1 of the notes (Assignments 2 and 3).
- A Notes sheet with Theorem 2.2, the Picard iterates, Euler's method, Abel's identity, and Corollary 3.1 is provided at the end of the quiz.
- I suggest you spend about one minute per point.

Problem 1 (12 points)

Consider the initial value problem

$$y'(t) = -4y(t), \quad y(0) = 2.$$

- Solve the initial value problem. Then apply Euler's method with step size $h > 0$, and show that $y_n = 2(1 - 4h)^n$ for every $n \geq 0$.
- For which step sizes $h > 0$ does $y_n \rightarrow 0$ as $n \rightarrow \infty$? For which step sizes $h > 0$ is $y_n > 0$ for every $n \geq 0$? Describe the behavior of the sequence $\{y_n\}_{n=0}^{\infty}$ for $h = 3/4$, and compare it with the behavior of the exact solution.
- The *implicit* Euler method reads $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$. Apply it to this problem, show that $y_n = 2(1 + 4h)^{-n}$ for every $n \geq 0$, and conclude that $y_n \rightarrow 0$ as $n \rightarrow \infty$ for every step size $h > 0$.

Problem 2 (13 points)

Consider the autonomous ODE

$$y'(t) = 4y(t) - y(t)^3.$$

- Find all equilibria, draw the phase line, and classify each equilibrium as stable or unstable.

- b) The ODE is a Bernoulli equation. Let $y_0 > 0$, and let y be the solution with $y(0) = y_0$. Use the substitution $v(t) = y(t)^{-2}$ to show that

$$y(t) = \frac{1}{\sqrt{\frac{1}{4} + (y_0^{-2} - \frac{1}{4}) e^{-8t}}}, \quad t \geq 0.$$

- c) Compute $\lim_{t \rightarrow \infty} y(t)$ from the formula in b), and check that your answer agrees with the phase line in a).

Problem 3 (12 points)

Theorem 2.2 (Picard–Lindelöf) and the definition of the Picard iterates are stated on the Notes sheet at the end of the quiz.

- a) Consider the initial value problem $y'(t) = 2\sqrt{|y(t)|}$, $y(0) = 0$. For every $c \geq 0$, show that the function

$$y_c(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq c, \\ (t - c)^2 & \text{if } t > c, \end{cases}$$

is a continuously differentiable solution on $[0, \infty)$. Conclude that the initial value problem has infinitely many solutions.

- b) Which of the assumptions (A1), (A2) and (A3) of Theorem 2.2 fail for the problem in a) on the rectangle $D_T = [0, T] \times [-C, C]$? Explain why a) does not contradict Theorem 2.2.
- c) Now consider the initial value problem $y'(t) = 3t^2(1 + y(t))$, $y(0) = 0$. Compute the Picard iterates φ_1 , φ_2 and φ_3 . Guess the limit of φ_k as $k \rightarrow \infty$, and check that it solves the initial value problem.

Problem 4 (13 points)

Consider the ODE

$$(1 + t)y''(t) + ty'(t) - y(t) = 0.$$

- a) Verify that $y_1(t) = t$ and $y_2(t) = e^{-t}$ are solutions for every $t \in \mathbb{R}$.
- b) Compute the Wronskian $W[y_1, y_2](t)$, and show that $\{y_1, y_2\}$ is a fundamental set of solutions on $I = (-1, \infty)$.
- c) Write the ODE in the form $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on $I = (-1, \infty)$, and check that your Wronskian satisfies Abel's identity.
- d) The Wronskian vanishes at $t = -1$ but at no other point. Explain why this does not contradict Corollary 3.1, which is stated on the Notes sheet at the end of the quiz.
- e) Find the solution on $I = (-1, \infty)$ that satisfies $y(0) = 1$ and $y'(0) = 1$.

Notes sheet

Theorem 2.2 (Picard–Lindelöf)

Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

Fix $C > 0$ and $T > t_0$, and let $D_T = \{(t, y) : t \in [t_0, T] \text{ and } y \in [y_0 - C, y_0 + C]\}$. Suppose that there exist constants $K, L > 0$ such that the following three assumptions hold:

(A1) The function f is continuous on D_T .

(A2) The function $t \mapsto f(t, y_0)$ satisfies $|f(t, y_0)| \leq K$ for every $t \in [t_0, T]$.

(A3) The function $y \mapsto f(t, y)$ is Lipschitz continuous, that is, $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ for every $t \in [t_0, T]$ and every $y_1, y_2 \in [y_0 - C, y_0 + C]$.

Suppose moreover that

$$C \geq \frac{K}{L} \left(e^{L(T-t_0)} - 1 \right).$$

Then there exists a unique solution y of the initial value problem that is continuously differentiable on $[t_0, T]$. Furthermore, $|y(t) - y_0| \leq C$ for every $t \in [t_0, T]$.

Picard iterates

Starting from the constant function $\varphi_0(t) = y_0$, the *Picard iterates* are

$$\varphi_{k+1}(t) = y_0 + \int_{t_0}^t f(\tau, \varphi_k(\tau)) d\tau, \quad k = 0, 1, 2, \dots$$

Under the assumptions of Theorem 2.2, the iterates converge on $[t_0, T]$ to the unique solution of the initial value problem.

Euler's method

Fix a step size $h > 0$ and let $t_n = t_0 + nh$. Starting from y_0 , Euler's method computes the approximations $y_n \approx y(t_n)$ by

$$y_{n+1} = y_n + hf(t_n, y_n), \quad n = 0, 1, 2, \dots$$

Proposition 3.3 (Abel's identity)

Let I be an open interval, let a_1 and a_0 be continuous on I , let y_1 and y_2 solve $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on I , and let $t_0 \in I$. Then the Wronskian $W = W[y_1, y_2]$ satisfies

$$W(t) = W(t_0)e^{-\int_{t_0}^t a_1(\tau) d\tau} \quad \text{for every } t \in I.$$

Corollary 3.1

Let I be an open interval, let a_1 and a_0 be continuous on I , and let y_1 and y_2 solve $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on I . Then the Wronskian $W = W[y_1, y_2]$ satisfies: either $W(t) = 0$ for every $t \in I$, or $W(t) \neq 0$ for every $t \in I$. In particular, y_1 and y_2 are linearly independent if and only if $W(t_*) \neq 0$ for some $t_* \in I$.