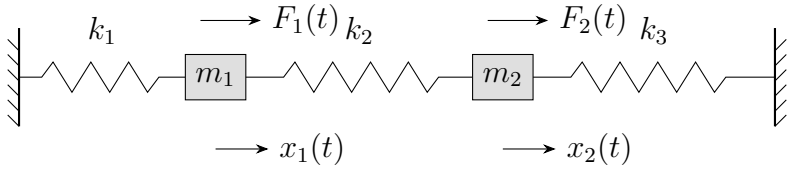
6 Linear systems of ODEs

We now turn to systems of n first-order linear ODEs,

$$\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{f}(t),$$

with unknown $\boldsymbol{y}(t) \in \mathbb{R}^n$, coefficient matrix $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$, and inhomogeneity $\boldsymbol{f}(t) \in \mathbb{R}^n$. Boldface lowercase letters denote vectors and boldface uppercase letters denote matrices. We develop the structure of the solution set once and in full (fundamental matrices, the Wronskian, the resolvent, and Duhamel's principle), prove the statements on nth-order equations deferred in Chapter 3, and solve constant-coefficient systems with eigenvalues and the matrix exponential. We close with phase portraits and the classification of equilibria of planar systems.

Example 6.1 (Coupled mass–spring system). Two masses m_1 and m_2 slide without friction between two walls. Three springs, with spring constants k_1, k_2, and k_3, connect the left wall to m_1, m_1 to m_2, and m_2 to the right wall. External forces $F_1(t)$ and $F_2(t)$ act on the masses, and $x_1(t)$ and $x_2(t)$ denote their displacements from equilibrium.



The middle spring is stretched by $x_2 - x_1$, so Hooke's law and Newton's second law give

$$\begin{aligned} m_1 x_1''(t) &= F_1(t) - k_1 x_1(t) + k_2\left(x_2(t) - x_1(t)\right), \\ m_2 x_2''(t) &= F_2(t) - k_2\left(x_2(t) - x_1(t)\right) - k_3 x_2(t). \end{aligned}$$

With the velocities $v_1 = x_1'$ and $v_2 = x_2'$, the two second-order equations become four first-order equations:

$$\begin{aligned} x_1'(t) &= v_1(t), \\ x_2'(t) &= v_2(t), \\ v_1'(t) &= \frac{1}{m_1}\left[F_1(t) - (k_1 + k_2)x_1(t) + k_2 x_2(t)\right], \\ v_2'(t) &= \frac{1}{m_2}\left[F_2(t) + k_2 x_1(t) - (k_2 + k_3)x_2(t)\right]. \end{aligned}$$

With $\boldsymbol{y} = (x_1, x_2, v_1, v_2)^\top$, this reads

$$\boldsymbol{y}'(t) = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{4\times 4}} \boldsymbol{y}(t) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ F_1(t)/m_1 \\ F_2(t)/m_2 \end{pmatrix}}_{\boldsymbol{f}(t)},$$

a linear system with constant coefficients and inhomogeneity $\boldsymbol{f}$.

Example 6.2 (Equations of order n). Remark 1.1 showed how to rewrite an nth-order ODE as a first-order system. For the linear equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(t), \tag{21}$$

we set $z_1 = y$, $z_2 = y'$, …, $z_n = y^{(n-1)}$, so that $z_1' = z_2$, $z_2' = z_3$, …, $z_{n-1}' = z_n$, and the equation itself gives $z_n' = y^{(n)} = f - a_{n-1}z_n - \cdots - a_0 z_1$. In matrix form,

$$\boldsymbol{z}'(t) = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & \cdots & \cdots & -a_{n-1}(t) \end{pmatrix}}_{\boldsymbol{C}(t)} \boldsymbol{z}(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}. \tag{22}$$

The matrix $\boldsymbol{C}(t)$ is called the *companion matrix* of (21). Every result on linear systems therefore applies to nth-order linear equations. For $n = 2$ and the harmonic oscillator, we recover the matrix $\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}$ of Section 1.3.

The predator–prey system of Example 1.4 is also a first-order system, but the products xy make it nonlinear. Chapter 7 treats nonlinear systems.

6.1 Linear algebra review

We use the following facts from linear algebra. A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has entries $(\boldsymbol{A})_{ij} = a_{ij}$, a vector $\boldsymbol{v} \in \mathbb{R}^n$ has components v_j, and $(\boldsymbol{Av})_i = \sum_{j=1}^n a_{ij}v_j$. Addition and scalar multiplication act entrywise, the product of $\boldsymbol{A}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ has entries $(\boldsymbol{AB})_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$, and matrix multiplication is associative but not commutative in general. The *trace* $\operatorname{tr}\boldsymbol{A} = a_{11} + \cdots + a_{nn}$ is the sum of the diagonal entries.

Determinants and inverses. The determinant depends linearly on each row and vanishes when two rows coincide. The matrix $\boldsymbol{A}$ is invertible if and only if $\det \boldsymbol{A} \neq 0$, if and only if its columns are linearly independent, and then $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ for every $\boldsymbol{b} \in \mathbb{R}^n$. For $n = 2$,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } ad - bc \neq 0.$$

Vector-valued functions. A function $\boldsymbol{y}\colon I \to \mathbb{R}^n$ on an interval I has components $y_1, \ldots, y_n \colon I \to \mathbb{R}$. It is continuous, differentiable, or integrable if each component is, and derivatives and integrals act componentwise: $\boldsymbol{y}'(t) = (y_1'(t), \ldots, y_n'(t))^\top$ and $\int_a^b \boldsymbol{y}(t)\,dt = \left(\int_a^b y_1(t)\,dt, \ldots, \int_a^b y_n(t)\,dt\right)^\top$. Matrix-valued functions $\boldsymbol{M}\colon I \to \mathbb{R}^{n\times n}$ follow the same conventions, and the product rule reads $(\boldsymbol{Mz})' = \boldsymbol{M}'\boldsymbol{z} + \boldsymbol{Mz}'$ and $(\boldsymbol{MN})' = \boldsymbol{M}'\boldsymbol{N} + \boldsymbol{MN}'$, in this order of the factors. Definitions 3.1 and 3.2 and Lemma 3.1 carry over verbatim to functions $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k \colon I \to \mathbb{R}^n$: they are linearly independent if and only if $\alpha_1\boldsymbol{y}_1(t) + \cdots + \alpha_k\boldsymbol{y}_k(t) = \boldsymbol{0}$ for every $t \in I$ forces $\alpha_1 = \cdots = \alpha_k = 0$.

Eigenvalues and eigenvectors. A number $\lambda \in \mathbb{C}$ is an *eigenvalue* of $\boldsymbol{A}$, with *eigenvector* $\boldsymbol{v}$, if $\boldsymbol{v} \in \mathbb{C}^n$ is nonzero and $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$, that is, $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$. Such a $\boldsymbol{v}$ exists if and only if $\boldsymbol{A} - \lambda \boldsymbol{I}$ is singular, that is, if and only if

$$p(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.$$

The *characteristic polynomial* p has degree n, so $\boldsymbol{A}$ has n eigenvalues in $\mathbb{C}$ counted with multiplicity. The eigenvectors for λ, together with $\boldsymbol{0}$, form the *eigenspace* $\ker(\boldsymbol{A} - \lambda \boldsymbol{I})$. A real eigenvalue has a real eigenvector. A nonreal eigenvalue λ comes with its conjugate $\bar{\lambda}$, with eigenvector $\bar{\boldsymbol{v}}$, because $\boldsymbol{A}$ is real and conjugating $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$ gives $\boldsymbol{A}\bar{\boldsymbol{v}} = \bar{\lambda}\bar{\boldsymbol{v}}$. For $n = 2$,

$$p(\lambda) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (\operatorname{tr} \boldsymbol{A})\lambda + \det \boldsymbol{A}.$$

Example 6.3. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Then $\operatorname{tr} \boldsymbol{A} = 5$ and $\det \boldsymbol{A} = 6$, so

$$p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) \quad \Longrightarrow \quad \lambda_1 = 2,\ \lambda_2 = 3.$$

For $\lambda_1 = 2$, the equation $(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$ reads

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad v_1 = 2v_2,$$

so the eigenvectors are the nonzero multiples of $\boldsymbol{v}_1 = (2, 1)^\top$. For $\lambda_2 = 3$, the equation $-2v_1 + 2v_2 = 0$ gives the nonzero multiples of $\boldsymbol{v}_2 = (1, 1)^\top$.

Diagonalization. Eigenvectors for distinct eigenvalues are linearly independent, so a matrix with n distinct eigenvalues has n linearly independent eigenvectors. Suppose, more generally, that $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, not necessarily distinct, and let $\boldsymbol{V} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \cdots\ \boldsymbol{v}_n]$. Then $\boldsymbol{V}$ is invertible, and the relations $\boldsymbol{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$ say that

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D}, \qquad \boldsymbol{D} = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

that is, $\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{D}$ and $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$: the matrix $\boldsymbol{A}$ is *diagonalizable*. When an eigenvalue is not real, $\boldsymbol{V}$ has complex entries and these identities hold in $\mathbb{C}^{n \times n}$. A matrix with a repeated eigenvalue may fail to be diagonalizable (Section 6.3).

6.2 General theory

Let $I \subseteq \mathbb{R}$ be an open interval, and let $\boldsymbol{A} \colon I \to \mathbb{R}^{n \times n}$ and $\boldsymbol{f} \colon I \to \mathbb{R}^n$ be continuous. We consider the system

$$\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{f}(t), \qquad t \in I, \tag{23}$$

together with its associated homogeneous system

$$\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t), \qquad t \in I. \tag{24}$$

A *solution* is a differentiable function $\boldsymbol{y}\colon I \to \mathbb{R}^n$ that satisfies the system at every $t \in I$. An IVP prescribes the value at a point $t_0 \in I$:

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0, \tag{25}$$

where $\boldsymbol{y}_0 \in \mathbb{R}^n$ is given. One vector condition replaces the n scalar conditions of an nth-order equation.

Theorem 6.1 (Existence and uniqueness for linear systems)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{f}$ be continuous on the open interval I, let $t_0 \in I$, and let $\boldsymbol{y}_0 \in \mathbb{R}^n$. Then the initial value problem* (23)–(25) *has exactly one solution on I.*

Proof. Deferred to Corollary 7.2, which deduces this theorem from the Picard–Lindelöf theorem for systems. On every compact subinterval of I, the right-hand side of (23) is Lipschitz continuous in $\boldsymbol{y}$ with a constant that depends only on the subinterval, which yields a solution on all of I rather than near t_0 only. □

Theorem 6.1 contains the existence and uniqueness theorem for nth-order linear equations: y solves (21) with prescribed $y(t_0), \dots, y^{(n-1)}(t_0)$ if and only if $\boldsymbol{z} = (y, y', \dots, y^{(n-1)})^{\top}$ solves the companion system (22) with $\boldsymbol{z}(t_0)$ prescribed, since the first $n-1$ equations of that system say that $z_{k+1} = z_1^{(k)}$. Theorem 3.1 is the case $n = 2$.

Proposition 6.1 (Superposition principle)**.** *If $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ solve the homogeneous system* (24) *and $c_1, c_2 \in \mathbb{R}$, then $c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2$ also solves* (24)*.*

Proof. Differentiation and multiplication by $\boldsymbol{A}(t)$ are linear, so $(c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2)' = c_1\boldsymbol{A}\boldsymbol{y}_1 + c_2\boldsymbol{A}\boldsymbol{y}_2 = \boldsymbol{A}(c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2)$ on I. □

Hence the solutions of (24) form a vector space S. The proof that a second-order equation has a two-dimensional solution space (Theorem 3.2) used only existence and uniqueness, and the same idea gives $\dim S$.

Theorem 6.2 (Structure of the solution set)**.** *Let $\boldsymbol{A}$ be continuous on I.*

(a) The solutions of the homogeneous system (24) *form a vector space S of dimension n.*

(b) If $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ are linearly independent solutions of (24)*, then every solution $\boldsymbol{y}$ of* (24) *can be written as $\boldsymbol{y} = c_1\boldsymbol{y}_1 + \cdots + c_n\boldsymbol{y}_n$ for unique constants $c_1, \dots, c_n \in \mathbb{R}$.*

Proof. (a) Fix $t_0 \in I$ and consider the linear evaluation map $E\colon S \to \mathbb{R}^n$, $E(\boldsymbol{y}) = \boldsymbol{y}(t_0)$. If $E(\boldsymbol{y}) = \boldsymbol{0}$, then $\boldsymbol{y}$ and the zero function solve the same initial value problem, so $\boldsymbol{y} \equiv 0$ by uniqueness (Theorem 6.1). Thus E is injective. For every $\boldsymbol{y}_0 \in \mathbb{R}^n$, existence gives a solution with $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, so E is surjective. A linear bijection preserves dimension, so $\dim S = n$.

(b) Any n linearly independent vectors in an n-dimensional space form a basis, so every $\boldsymbol{y} \in S$ has a unique expansion in $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$. □

Definition 6.1. A set $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_n\}$ of n linearly independent solutions of (24) is called a *fundamental set of solutions*, the matrix-valued function

$$\boldsymbol{\Phi}(t) = [\boldsymbol{y}_1(t)\ \boldsymbol{y}_2(t)\ \cdots\ \boldsymbol{y}_n(t)] \in \mathbb{R}^{n\times n}, \qquad t \in I,$$

whose columns are these solutions is called a *fundamental matrix*, and the expression $c_1\boldsymbol{y}_1 + \cdots + c_n\boldsymbol{y}_n = \boldsymbol{\Phi}\boldsymbol{c}$ with $\boldsymbol{c} = (c_1, \dots, c_n)^\top \in \mathbb{R}^n$ arbitrary is called the *general solution* of (24). The *Wronskian* of n solutions $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ of (24) is the function $W(t) = \det[\boldsymbol{y}_1(t)\ \cdots\ \boldsymbol{y}_n(t)]$, $t \in I$.

Each column of $\boldsymbol{\Phi}$ solves (24), so a fundamental matrix satisfies the *matrix differential equation*

$$\boldsymbol{\Phi}'(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t), \qquad t \in I.$$

Conversely, the columns of any solution $\boldsymbol{\Phi}\colon I \to \mathbb{R}^{n\times n}$ of this equation solve (24), and by the next proposition they form a fundamental set if and only if $\det\boldsymbol{\Phi} \neq 0$.

Proposition 6.2 (Wronskian test)**.** *Let $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ solve* (24) *on I, with Wronskian W. The following statements are equivalent:*

(a) $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ are linearly independent;

(b) $W(t) \neq 0$ for every $t \in I$;

(c) $W(t_*) \neq 0$ for some $t_* \in I$.

Proof. Write $\boldsymbol{\Phi}(t) = [\boldsymbol{y}_1(t)\ \cdots\ \boldsymbol{y}_n(t)]$, so that $\alpha_1\boldsymbol{y}_1(t) + \cdots + \alpha_n\boldsymbol{y}_n(t) = \boldsymbol{\Phi}(t)\boldsymbol{\alpha}$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\top$.

(b) $\Longrightarrow$ (c) is clear.

(c) $\Longrightarrow$ (a): If $\boldsymbol{\Phi}(t)\boldsymbol{\alpha} = \boldsymbol{0}$ for every $t \in I$, then at $t = t_*$ the invertible matrix $\boldsymbol{\Phi}(t_*)$ gives $\boldsymbol{\alpha} = \boldsymbol{0}$, so $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ are linearly independent.

(a) $\Longrightarrow$ (b): Suppose that $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ are linearly independent but $W(t_*) = 0$ for some $t_* \in I$. Then $\boldsymbol{\Phi}(t_*)$ is singular, so $\boldsymbol{\Phi}(t_*)\tilde{\boldsymbol{c}} = \boldsymbol{0}$ for some $\tilde{\boldsymbol{c}} \neq \boldsymbol{0}$. The function $\boldsymbol{y} = \boldsymbol{\Phi}\tilde{\boldsymbol{c}}$ solves (24) by superposition and vanishes at t_*, as does the zero function, so uniqueness (Theorem 6.1) forces $\boldsymbol{y} \equiv 0$ on I. Thus $\tilde{c}_1\boldsymbol{y}_1 + \cdots + \tilde{c}_n\boldsymbol{y}_n = 0$ on I with $\tilde{\boldsymbol{c}} \neq \boldsymbol{0}$, contradicting linear independence. □

By Proposition 6.2, the Wronskian of n solutions vanishes either identically or nowhere. The next identity makes this explicit and extends Abel's identity (Proposition 3.3), which Chapter 3 stated without proof for order n.

Proposition 6.3 (Abel–Liouville identity)**.** *Let $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ solve* (24) *on I, with Wronskian W, and let $t_0 \in I$. Then $W'(t) = \operatorname{tr}\boldsymbol{A}(t)W(t)$ for every $t \in I$, and therefore*

$$W(t) = W(t_0)e^{\int_{t_0}^{t} \operatorname{tr}\boldsymbol{A}(\tau)\,d\tau} \qquad \text{for every } t \in I.$$

Proof. Let $\boldsymbol{\Phi} = [\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_n]$ have rows $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n$. The determinant is a sum of products of n entries, one from each row, so the product rule gives

$$W'(t) = \sum_{i=1}^{n} \det \boldsymbol{\Phi}_i(t),$$

where $\boldsymbol{\Phi}_i$ is $\boldsymbol{\Phi}$ with its ith row replaced by $\boldsymbol{r}_i'$. The ith row of $\boldsymbol{\Phi}' = \boldsymbol{A}\boldsymbol{\Phi}$ reads $\boldsymbol{r}_i' = \sum_{k=1}^{n} a_{ik}\boldsymbol{r}_k$, and the determinant is linear in the ith row, so

$$\det \boldsymbol{\Phi}_i = \sum_{k=1}^{n} a_{ik} \det\left(\boldsymbol{\Phi} \text{ with row } i \text{ replaced by } \boldsymbol{r}_k\right) = a_{ii} \det \boldsymbol{\Phi},$$

because for $k \neq i$ the matrix has two equal rows. Summing over i gives $W' = \operatorname{tr}\boldsymbol{A} \cdot W$, a first-order linear homogeneous ODE, whose solutions are $W(t) = W(t_0)e^{\int_{t_0}^{t} \operatorname{tr}\boldsymbol{A}(\tau)\,d\tau}$ by Section 2.1. □

Example 6.4. The functions $y_1(t) = t$ and $y_2(t) = t^2$ solve $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$ on $I = (0, \infty)$, whose companion system (22) has

$$\boldsymbol{C}(t) = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}, \qquad \operatorname{tr}\boldsymbol{C}(t) = \frac{2}{t}.$$

The solutions $\boldsymbol{y}_1 = (t, 1)^\top$ and $\boldsymbol{y}_2 = (t^2, 2t)^\top$ have Wronskian $W(t) = t \cdot 2t - 1 \cdot t^2 = t^2$, and Proposition 6.3 with $t_0 = 1$ predicts $W(t) = W(1)e^{\int_1^t 2/\tau\,d\tau} = e^{2\ln t} = t^2$. ✓

6.2.1 The resolvent

By Theorem 6.2, every solution of (24) has the form $\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{c}$. The initial condition $\boldsymbol{\Phi}(t_0)\boldsymbol{c} = \boldsymbol{y}_0$ gives $\boldsymbol{c} = \boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{y}_0$, since $\boldsymbol{\Phi}(t_0)$ is invertible by Proposition 6.2, and

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{y}_0.$$

The solution depends linearly on $\boldsymbol{y}_0$, and we name the matrix in front of $\boldsymbol{y}_0$.

Definition 6.2. Let $\boldsymbol{\Phi}$ be a fundamental matrix of (24). The *resolvent* (or *transition matrix*) of (24) is

$$\boldsymbol{R}(t, s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}, \qquad t, s \in I.$$

Proposition 6.4 (Properties of the resolvent)**.** *Let $\boldsymbol{A}$ be continuous on I.*

(a) For every $s \in I$ and $\boldsymbol{y}_0 \in \mathbb{R}^n$, the function $t \mapsto \boldsymbol{R}(t, s)\boldsymbol{y}_0$ is the solution of (24) with $\boldsymbol{y}(s) = \boldsymbol{y}_0$. In particular, $\boldsymbol{R}(s, s) = \boldsymbol{I}$ and $\frac{\partial}{\partial t}\boldsymbol{R}(t, s) = \boldsymbol{A}(t)\boldsymbol{R}(t, s)$.

(b) The resolvent does not depend on the choice of the fundamental matrix $\boldsymbol{\Phi}$.

(c) For all $r, s, t \in I$, $\boldsymbol{R}(t, s)\boldsymbol{R}(s, r) = \boldsymbol{R}(t, r)$, and $\boldsymbol{R}(t, s)$ is invertible with $\boldsymbol{R}(t, s)^{-1} = \boldsymbol{R}(s, t)$.

Proof. (a) The function $\boldsymbol{y} = \boldsymbol{\Phi c}$ with $\boldsymbol{c} = \boldsymbol{\Phi}(s)^{-1}\boldsymbol{y}_0$ solves (24) by superposition, and $\boldsymbol{y}(s) = \boldsymbol{y}_0$. With $\boldsymbol{y}_0$ each standard basis vector, this gives $\boldsymbol{R}(s,s) = \boldsymbol{I}$ and $\frac{\partial}{\partial t}\boldsymbol{R}(t,s) = \boldsymbol{A}(t)\boldsymbol{R}(t,s)$ column by column.

(b) By (a) and uniqueness, $\boldsymbol{R}(t,s)\boldsymbol{y}_0$ is the value at t of the unique solution equal to $\boldsymbol{y}_0$ at s, whichever fundamental matrix defines $\boldsymbol{R}$, and two matrices that agree on every vector are equal.

(c) Let $\boldsymbol{y}$ be the solution with $\boldsymbol{y}(r) = \boldsymbol{y}_0$. By (a), $\boldsymbol{y}(s) = \boldsymbol{R}(s,r)\boldsymbol{y}_0$, and by (a) again with initial time s, $\boldsymbol{y}(t) = \boldsymbol{R}(t,s)\boldsymbol{y}(s) = \boldsymbol{R}(t,s)\boldsymbol{R}(s,r)\boldsymbol{y}_0$. Also $\boldsymbol{y}(t) = \boldsymbol{R}(t,r)\boldsymbol{y}_0$, and $\boldsymbol{y}_0$ is arbitrary, so $\boldsymbol{R}(t,s)\boldsymbol{R}(s,r) = \boldsymbol{R}(t,r)$. With $r = t$, this gives $\boldsymbol{R}(t,s)\boldsymbol{R}(s,t) = \boldsymbol{I}$. □

Property (c) says that flowing from r to s and then to t equals flowing from r to t directly. In particular, solutions exist backward in time as well as forward.

6.2.2 Inhomogeneous systems

If $\boldsymbol{y}$ and $\boldsymbol{y}_p$ both solve (23), then $(\boldsymbol{y}-\boldsymbol{y}_p)' = \boldsymbol{A}(\boldsymbol{y}-\boldsymbol{y}_p)$, so, as in Theorem 3.3, every solution of (23) is one particular solution $\boldsymbol{y}_p$ plus a homogeneous solution $\boldsymbol{\Phi c}$. As in Section 3.3.2, we find $\boldsymbol{y}_p$ by varying the constants.

Theorem 6.3 (Variation of constants; Duhamel's principle)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{f}$ be continuous on I, let $t_0 \in I$ and $\boldsymbol{y}_0 \in \mathbb{R}^n$, and let $\boldsymbol{R}$ be the resolvent of the homogeneous system* (24)*. Then the solution of the initial value problem* (23)–(25) *is*

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{R}(t,t_0)\boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{R}(t,s)\boldsymbol{f}(s)\,ds, \qquad t \in I.}$$

Proof. Let $\boldsymbol{\Phi}$ be a fundamental matrix of (24), and try $\boldsymbol{y} = \boldsymbol{\Phi c}$ with a differentiable $\boldsymbol{c}\colon I \to \mathbb{R}^n$ in place of the constant vector. The product rule and $\boldsymbol{\Phi}' = \boldsymbol{A\Phi}$ give

$$\boldsymbol{y}' = \boldsymbol{\Phi}'\boldsymbol{c} + \boldsymbol{\Phi c}' = \boldsymbol{A\Phi c} + \boldsymbol{\Phi c}' = \boldsymbol{Ay} + \boldsymbol{\Phi c}',$$

so $\boldsymbol{y}$ solves (23) if and only if $\boldsymbol{\Phi c}' = \boldsymbol{f}$, that is,

$$\boldsymbol{c}'(t) = \boldsymbol{\Phi}(t)^{-1}\boldsymbol{f}(t).$$

Unlike in Section 3.3.2, the first-order system needs no extra condition. The right-hand side is continuous, so $\boldsymbol{c}(t) = \boldsymbol{c}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(s)^{-1}\boldsymbol{f}(s)\,ds$, and $\boldsymbol{y}(t_0) = \boldsymbol{\Phi}(t_0)\boldsymbol{c}(t_0) = \boldsymbol{y}_0$ gives $\boldsymbol{c}(t_0) = \boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{y}_0$. Therefore,

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}\boldsymbol{f}(s)\,ds,$$

which is the claimed formula, since $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1} = \boldsymbol{R}(t,s)$ and $\boldsymbol{\Phi}(t)$ does not depend on s. This $\boldsymbol{y}$ solves the initial value problem by construction and is its only solution by Theorem 6.1. □

The formula splits the solution into the response $\boldsymbol{R}(t,t_0)\boldsymbol{y}_0$ to the initial value and the response $\int_{t_0}^{t}\boldsymbol{R}(t,s)\boldsymbol{f}(s)\,ds$ to the forcing. The integrand is the state at time t produced by the impulse $\boldsymbol{f}(s)\,ds$ at time s, and the integral adds these contributions: this is *Duhamel's principle*.

Remark 6.1 (Green's functions, revisited). Section 4.6 wrote the solution of an nth-order equation with constant coefficients and zero initial values as $y = g * f$, with Green's function $g = \mathcal{L}^{-1}[1/p]$. Theorem 6.3 explains where g comes from. The companion system (22) has constant matrix $\boldsymbol{C}$ and forcing $f(t)\boldsymbol{e}_n$, where $\boldsymbol{e}_n = (0,\dots,0,1)^\top$, and its resolvent depends only on $t-s$ (Section 6.3). Write $\boldsymbol{R}(t,s) = \boldsymbol{R}(t-s)$. With $t_0 = 0$ and $\boldsymbol{z}_0 = \boldsymbol{0}$, the first component of Duhamel's formula is

$$y(t) = z_1(t) = \int_0^t \left(\boldsymbol{R}(t-s)\boldsymbol{e}_n\right)_1 f(s)\,ds = \int_0^t g(t-s)f(s)\,ds, \qquad g(t) := \left(\boldsymbol{R}(t)\boldsymbol{e}_n\right)_1.$$

So the Green's function is the first component of the solution with initial value $\boldsymbol{e}_n$: the solution of the homogeneous equation with $y(0) = \cdots = y^{(n-2)}(0) = 0$ and $y^{(n-1)}(0) = 1$, the response to a unit impulse at $t = 0$ (Section 4.6). For $n = 2$ and variable coefficients, the companion system has fundamental matrix $\boldsymbol{\Phi} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, and the $(1,2)$ entry of $\boldsymbol{R}(t,s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}$ is

$$\frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(s)},$$

the kernel of the variation-of-constants formula of Section 3.3.2.

6.2.3 Equations of order n

We now prove the statements on nth-order linear equations that Chapter 3 left open. Consider the homogeneous equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0, \qquad t \in I, \tag{26}$$

with $a_0,\dots,a_{n-1}$ continuous on I, and its companion system $\boldsymbol{z}' = \boldsymbol{C}(t)\boldsymbol{z}$ from (22). For $n-1$ times differentiable functions $y_1,\dots,y_n$ on I, the *Wronskian* is

$$W[y_1,\dots,y_n](t) = \det\begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix},$$

which reduces to Definition 3.3 when $n = 2$.

Theorem 6.4 (Linear equations of order n)**.** *Let $a_0,\dots,a_{n-1}$ be continuous on I.*

(a) The solutions of (26) *form a vector space of dimension n. If $y_1,\dots,y_n$ are linearly independent solutions, then every solution has the form $c_1y_1 + \cdots + c_ny_n$ for unique constants $c_1,\dots,c_n$.*

(b) Solutions $y_1, \dots, y_n$ of (26) are linearly independent if and only if $W[y_1, \dots, y_n](t) \neq 0$ for every $t \in I$, if and only if $W[y_1, \dots, y_n](t_*) \neq 0$ for some $t_* \in I$.

(c) (Abel's identity) For solutions $y_1, \dots, y_n$ of (26) and $t_0 \in I$,

$$W[y_1, \dots, y_n](t) = W[y_1, \dots, y_n](t_0) e^{-\int_{t_0}^{t} a_{n-1}(\tau)\, d\tau} \qquad \text{for every } t \in I.$$

Proof. The linear map $y \mapsto \boldsymbol{z} = (y, y', \dots, y^{(n-1)})^\top$ sends solutions of (26) to solutions of the companion system. It is injective, since $z_1 = y$, and surjective by the discussion after Theorem 6.1. Hence the two solution spaces are isomorphic, and (a) follows from Theorem 6.2. Moreover, $\sum_j \alpha_j y_j \equiv 0$ if and only if $\sum_j \alpha_j \boldsymbol{z}_j \equiv 0$ (differentiate the first identity $n-1$ times, or take the first component of the second), so the y_j are linearly independent if and only if the $\boldsymbol{z}_j$ are. The matrix $[\boldsymbol{z}_1 \ \cdots \ \boldsymbol{z}_n]$ is the matrix in the definition of $W[y_1, \dots, y_n]$, so the two Wronskians coincide and (b) is Proposition 6.2. Finally, the diagonal of $\boldsymbol{C}(t)$ is $(0, \dots, 0, -a_{n-1}(t))$, so $\operatorname{tr} \boldsymbol{C} = -a_{n-1}$ and (c) is Proposition 6.3. □

6.3 Constant coefficients

In this section, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is constant, so Section 6.2 applies with $I = \mathbb{R}$. We consider

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t), \qquad t \in \mathbb{R}. \tag{27}$$

For $n = 1$, the solutions are the exponentials ce^{at}, and for second-order equations the Ansatz $y(t) = e^{\lambda t}$ led to the characteristic polynomial (Section 3.2). We use the same Ansatz for systems.

6.3.1 The eigenvalue Ansatz and diagonalization

Try $\boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{v}$ with a number λ and a constant nonzero vector $\boldsymbol{v}$. Then $\boldsymbol{y}'(t) = \lambda e^{\lambda t}\boldsymbol{v}$ and $\boldsymbol{A}\boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{A}\boldsymbol{v}$, and the exponential never vanishes, so

$$e^{\lambda t}\boldsymbol{v} \text{ solves (27)} \quad \Longleftrightarrow \quad \boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

The Ansatz succeeds exactly when λ is an eigenvalue and $\boldsymbol{v}$ an eigenvector of $\boldsymbol{A}$. Each eigenvector gives one solution, and we need n linearly independent ones.

Theorem 6.5. *Suppose that $\boldsymbol{A}$ has n linearly independent real eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ with real eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\{e^{\lambda_1 t}\boldsymbol{v}_1, \dots, e^{\lambda_n t}\boldsymbol{v}_n\}$ is a fundamental set of solutions of (27), and the general solution is*

$$\boxed{\boldsymbol{y}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{v}_n = \boldsymbol{V} \operatorname{diag}\left(e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right) \boldsymbol{c}, \qquad \boldsymbol{c} \in \mathbb{R}^n,}$$

where $\boldsymbol{V} = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$. The solution with $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ has $\boldsymbol{c} = \operatorname{diag}(e^{-\lambda_1 t_0}, \dots, e^{-\lambda_n t_0})\boldsymbol{V}^{-1}\boldsymbol{y}_0$.

Proof. Each $e^{\lambda_j t}\boldsymbol{v}_j$ solves (27) by the computation above, and their Wronskian at $t = 0$ is $\det \boldsymbol{V} \neq 0$, because the columns of $\boldsymbol{V}$ are linearly independent. By Proposition 6.2, the n solutions are linearly independent, and Theorem 6.2 gives the general solution. The matrix form writes $\sum_j c_j e^{\lambda_j t}\boldsymbol{v}_j$ as $\boldsymbol{V}$ times the vector with components $e^{\lambda_j t}c_j$, and $\boldsymbol{y}(t_0) = \boldsymbol{V} \operatorname{diag}(e^{\lambda_j t_0})\boldsymbol{c} = \boldsymbol{y}_0$ gives the stated $\boldsymbol{c}$. □

With $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$ and $\boldsymbol{z} = \boldsymbol{V}^{-1}\boldsymbol{y}$, the system becomes $\boldsymbol{z}' = \boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V}\boldsymbol{z} = \boldsymbol{D}\boldsymbol{z}$, that is, $z_j' = \lambda_j z_j$ for $j = 1, \ldots, n$. The change of variables *decouples* the system into n scalar equations with solutions $z_j(t) = c_j e^{\lambda_j t}$, and $\boldsymbol{y} = \boldsymbol{V}\boldsymbol{z}$ recovers the boxed formula.

Example 6.5. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $\boldsymbol{y}(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. By Example 6.3, $\lambda_1 = 2$ with $\boldsymbol{v}_1 = (2,1)^\top$ and $\lambda_2 = 3$ with $\boldsymbol{v}_2 = (1,1)^\top$, so the general solution is

$$\boldsymbol{y}(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The initial condition reads $\boldsymbol{V}\boldsymbol{c} = \boldsymbol{y}(0)$, so

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \boldsymbol{V}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and therefore

$$\boldsymbol{y}(t) = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{2t} + 2e^{3t} \\ e^{2t} + 2e^{3t} \end{pmatrix}.$$

Complex eigenvalues. Let $\lambda = \sigma + i\tau$ with $\tau \neq 0$ be an eigenvalue of $\boldsymbol{A}$, with eigenvector $\boldsymbol{v} = \boldsymbol{a} + i\boldsymbol{b}$, where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. The Ansatz still works over $\mathbb{C}$: $e^{\lambda t}\boldsymbol{v}$ solves (27), and so does its conjugate $e^{\bar{\lambda} t}\bar{\boldsymbol{v}}$. Since $\boldsymbol{A}$ is real, the real and imaginary parts of a complex solution are real solutions (take real and imaginary parts of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$). Euler's formula gives

$$\begin{aligned} e^{\lambda t}\boldsymbol{v} &= e^{\sigma t}\left(\cos(\tau t) + i\sin(\tau t)\right)(\boldsymbol{a} + i\boldsymbol{b}) \\ &= e^{\sigma t}\left(\boldsymbol{a}\cos(\tau t) - \boldsymbol{b}\sin(\tau t)\right) + ie^{\sigma t}\left(\boldsymbol{a}\sin(\tau t) + \boldsymbol{b}\cos(\tau t)\right), \end{aligned}$$

so the pair $\lambda, \bar{\lambda}$ contributes the two real solutions

$$\boxed{\boldsymbol{y}_1(t) = e^{\sigma t}\left(\boldsymbol{a}\cos(\tau t) - \boldsymbol{b}\sin(\tau t)\right), \qquad \boldsymbol{y}_2(t) = e^{\sigma t}\left(\boldsymbol{a}\sin(\tau t) + \boldsymbol{b}\cos(\tau t)\right).}$$

They are linearly independent, because $\boldsymbol{y}_1(0) = \boldsymbol{a}$ and $\boldsymbol{y}_2(0) = \boldsymbol{b}$ are. The eigenvectors $\boldsymbol{v} = \boldsymbol{a} + i\boldsymbol{b}$ and $\bar{\boldsymbol{v}} = \boldsymbol{a} - i\boldsymbol{b}$ belong to distinct eigenvalues, so they are linearly independent over $\mathbb{C}$, and hence $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent over $\mathbb{R}$. When $\boldsymbol{A}$ has n linearly independent eigenvectors, some complex, replacing each conjugate pair $e^{\lambda t}\boldsymbol{v}, e^{\bar{\lambda} t}\bar{\boldsymbol{v}}$ by the real pair $\boldsymbol{y}_1, \boldsymbol{y}_2$ yields a real fundamental set.

Example 6.6. Consider $x' = x - y$, $y' = x + y$, that is, $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ with $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Here $\operatorname{tr}\boldsymbol{A} = 2$ and $\det\boldsymbol{A} = 2$, so $p(\lambda) = \lambda^2 - 2\lambda + 2$ and $\lambda = 1 \pm i$. For $\lambda = 1 + i$, the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$ has first row $-iv_1 - v_2 = 0$, so $\boldsymbol{v} = (i, 1)^\top = \boldsymbol{a} + i\boldsymbol{b}$ with $\boldsymbol{a} = (0,1)^\top$ and $\boldsymbol{b} = (1,0)^\top$. With $\sigma = \tau = 1$, the boxed formulas give

$$\boldsymbol{y}_1(t) = e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \qquad \boldsymbol{y}_2(t) = e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \qquad \boldsymbol{y}(t) = e^t \begin{pmatrix} c_2\cos t - c_1\sin t \\ c_1\cos t + c_2\sin t \end{pmatrix}.$$

Every nonzero solution rotates counterclockwise around the origin while its norm $e^t\sqrt{c_1^2 + c_2^2}$ grows exponentially.

Repeated eigenvalues. A repeated eigenvalue may leave $\boldsymbol{A}$ with fewer than n linearly independent eigenvectors, and then the Ansatz produces too few solutions. Let $n = 2$ and $p(\lambda) = (\lambda - \lambda_0)^2$, so the double root $\lambda_0 = \frac{1}{2}\operatorname{tr}\boldsymbol{A}$ is real. Every 2×2 matrix satisfies $\boldsymbol{A}^2 - (\operatorname{tr}\boldsymbol{A})\boldsymbol{A} + (\det\boldsymbol{A})\boldsymbol{I} = \boldsymbol{0}$ (the Cayley–Hamilton theorem; check it directly), so here

$$(\boldsymbol{A} - \lambda_0\boldsymbol{I})^2 = \boldsymbol{0}.$$

If the eigenspace of λ_0 is all of $\mathbb{R}^2$, then $\boldsymbol{A} = \lambda_0\boldsymbol{I}$ and the general solution is $\boldsymbol{y}(t) = e^{\lambda_0 t}\boldsymbol{c}$. Otherwise, the eigenspace is a line. Choose any vector $\boldsymbol{w}$ outside it. Then $\boldsymbol{v} := (\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{w} \neq \boldsymbol{0}$ and $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{v} = (\boldsymbol{A} - \lambda_0\boldsymbol{I})^2\boldsymbol{w} = \boldsymbol{0}$, so $\boldsymbol{v}$ is an eigenvector, and $\boldsymbol{w}$, which satisfies $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}$, is called a *generalized eigenvector*. As in Case 3 of Section 3.2, we try a solution with a factor t:

$$\boldsymbol{y}_2(t) = e^{\lambda_0 t}\,(t\boldsymbol{v} + \boldsymbol{w})\,.$$

Then $\boldsymbol{y}_2'(t) = e^{\lambda_0 t}\,(\lambda_0 t\boldsymbol{v} + \lambda_0\boldsymbol{w} + \boldsymbol{v})$ and $\boldsymbol{A}\boldsymbol{y}_2(t) = e^{\lambda_0 t}\,(t\lambda_0\boldsymbol{v} + \lambda_0\boldsymbol{w} + \boldsymbol{v})$, using $\boldsymbol{A}\boldsymbol{v} = \lambda_0\boldsymbol{v}$ and $\boldsymbol{A}\boldsymbol{w} = \lambda_0\boldsymbol{w} + \boldsymbol{v}$. Hence $\boldsymbol{y}_2$ solves (27), as does $\boldsymbol{y}_1(t) = e^{\lambda_0 t}\boldsymbol{v}$, and their Wronskian at $t = 0$ is $\det[\boldsymbol{v}\ \boldsymbol{w}] \neq 0$, because $\boldsymbol{w}$ is not a multiple of $\boldsymbol{v}$. The general solution is

$$\boxed{\boldsymbol{y}(t) = e^{\lambda_0 t}\,(c_1\boldsymbol{v} + c_2(t\boldsymbol{v} + \boldsymbol{w}))\,.}$$

Example 6.7. Let $\boldsymbol{A} = \begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix}$ and $\boldsymbol{y}(0) = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$. Here $\operatorname{tr}\boldsymbol{A} = 0$ and $\det\boldsymbol{A} = -25 + 25 = 0$, so $p(\lambda) = \lambda^2$ and $\lambda_0 = 0$ is a double eigenvalue. The eigenvectors solve $-5v_1 + v_2 = 0$, so the eigenspace is the line spanned by $(1, 5)^\top$, and $\boldsymbol{A}$ is not diagonalizable. Choose $\boldsymbol{w} = (0, 1)^\top$ outside the eigenspace. Then $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{w} = (1, 5)^\top$, and the general solution is

$$\boldsymbol{y}(t) = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\left(t\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

The initial condition gives $c_1 = -3$ and $5c_1 + c_2 = 5$, so $c_2 = 20$ and $\boldsymbol{y}(t) = \begin{pmatrix} 20t - 3 \\ 100t + 5 \end{pmatrix}$.

Remark 6.2 (General n). For an $n \times n$ matrix with an eigenvalue λ of multiplicity m whose eigenspace has dimension less than m, the missing solutions have the form

$$e^{\lambda t}\left(\boldsymbol{w}_k + t\boldsymbol{w}_{k-1} + \frac{t^2}{2!}\boldsymbol{w}_{k-2} + \cdots + \frac{t^k}{k!}\boldsymbol{w}_0\right),$$

where $\boldsymbol{w}_0$ is an eigenvector and $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{w}_j = \boldsymbol{w}_{j-1}$ for $j = 1, \ldots, k$ (a *chain* of generalized eigenvectors). By the Jordan normal form theorem, such chains provide n linearly independent solutions for every $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. We omit the proof. Compare the functions $t^k e^{\lambda t}$ of Remark 3.1.

Remark 6.3 (The companion matrix). For the nth-order equation with constant coefficients, expanding $\det(\lambda\boldsymbol{I} - \boldsymbol{C})$ along the first column and inducting on n gives $\det(\lambda\boldsymbol{I} - \boldsymbol{C}) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$, the characteristic polynomial p of Remark 3.1. For a root λ of p, the vector $\boldsymbol{v} = (1, \lambda, \ldots, \lambda^{n-1})^\top$ satisfies $\boldsymbol{C}\boldsymbol{v} = \lambda\boldsymbol{v}$, and the solution $e^{\lambda t}\boldsymbol{v}$ has first component $e^{\lambda t}$. The eigenvalue Ansatz thus contains the Ansatz $y = e^{\lambda t}$ of Chapter 3.

6.3.2 The matrix exponential

The scalar equation $y' = ay$ has the solution $y(t) = e^{at}y_0$. Does $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ have the solution $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$, and what is the exponential of a matrix? Since $e^x = \sum_{k=0}^{\infty} x^k/k!$, we define it by the same series.

Definition 6.3. For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the *matrix exponential* is

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \cdots \in \mathbb{R}^{n\times n},$$

where $\boldsymbol{A}^0 = \boldsymbol{I}$ and $\boldsymbol{A}^k = \boldsymbol{A}\cdot\boldsymbol{A}\cdots\boldsymbol{A}$ (k factors).

The series converges absolutely in norm, hence entrywise. Indeed, the operator norm $\|\boldsymbol{A}\| = \max_{\|\boldsymbol{v}\|=1} \|\boldsymbol{A}\boldsymbol{v}\|$ satisfies $\|\boldsymbol{A}\boldsymbol{v}\| \le \|\boldsymbol{A}\|\,\|\boldsymbol{v}\|$ and $\|\boldsymbol{A}\boldsymbol{B}\| \le \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|$, so $\left\|\boldsymbol{A}^k/k!\right\| \le \|\boldsymbol{A}\|^k/k!$, and $\sum_k \|\boldsymbol{A}\|^k/k! = e^{\|\boldsymbol{A}\|} < \infty$. The same bound shows that each entry of $e^{\boldsymbol{A}t} = \sum_k t^k\boldsymbol{A}^k/k!$ is a power series in t with infinite radius of convergence, which we may differentiate term by term (Section 5.1).

Theorem 6.6. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$ for $t \in \mathbb{R}$. Then $\boldsymbol{\Phi}'(t) = \boldsymbol{A}\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}(0) = \boldsymbol{I}$. Hence $e^{\boldsymbol{A}t}$ is a fundamental matrix of (27).*

Proof. Term-by-term differentiation gives

$$\boldsymbol{\Phi}'(t) = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^k\boldsymbol{A}^k}{k!} = \sum_{k=1}^{\infty}\frac{kt^{k-1}\boldsymbol{A}^k}{k!} = \boldsymbol{A}\sum_{k=1}^{\infty}\frac{t^{k-1}\boldsymbol{A}^{k-1}}{(k-1)!} = \boldsymbol{A}\sum_{k=0}^{\infty}\frac{t^k\boldsymbol{A}^k}{k!} = \boldsymbol{A}\boldsymbol{\Phi}(t),$$

and $\boldsymbol{\Phi}(0) = \boldsymbol{A}^0 = \boldsymbol{I}$. Each column of $\boldsymbol{\Phi}$ therefore solves (27), and $\det\boldsymbol{\Phi}(0) = 1$, so the columns form a fundamental set by Proposition 6.2. □

Proposition 6.5 (Properties of the matrix exponential)**.** *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$.*

(a) If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, then $e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}}$. (The conclusion may fail for noncommuting matrices.)

(b) $e^{\boldsymbol{A}}$ is invertible, with $(e^{\boldsymbol{A}})^{-1} = e^{-\boldsymbol{A}}$; in particular, $e^{\boldsymbol{A}t}e^{\boldsymbol{A}s} = e^{\boldsymbol{A}(t+s)}$ for all $t, s \in \mathbb{R}$.

(c) If $\boldsymbol{V}$ is invertible, then $e^{\boldsymbol{V}\boldsymbol{B}\boldsymbol{V}^{-1}} = \boldsymbol{V}e^{\boldsymbol{B}}\boldsymbol{V}^{-1}$. If $\boldsymbol{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^{\boldsymbol{D}} = \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.

(d) In general, $(e^{\boldsymbol{A}})_{ij} \neq e^{a_{ij}}$: the exponential is not computed entrywise.

Proof. (a) The matrix $\boldsymbol{B}$ commutes with every power of $\boldsymbol{A}$, hence with the partial sums of $e^{\boldsymbol{A}t}$, hence with $e^{\boldsymbol{A}t}$. Let $\boldsymbol{X}(t) = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}$. By Theorem 6.6 and the product rule,

$$\boldsymbol{X}'(t) = \boldsymbol{A}e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} + e^{\boldsymbol{A}t}\boldsymbol{B}e^{\boldsymbol{B}t} = \boldsymbol{A}e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} + \boldsymbol{B}e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} = (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}(t),$$

and $\boldsymbol{X}(0) = \boldsymbol{I}$. By Theorem 6.6, $e^{(\boldsymbol{A}+\boldsymbol{B})t}$ solves the same matrix initial value problem, so uniqueness (Theorem 6.1, column by column, for the coefficient matrix $\boldsymbol{A}+\boldsymbol{B}$) gives $\boldsymbol{X}(t) = e^{(\boldsymbol{A}+\boldsymbol{B})t}$ for every t. At $t = 1$, this is the claim.

(b) Apply (a) with $\boldsymbol{B} = -\boldsymbol{A}$: $e^{\boldsymbol{A}}e^{-\boldsymbol{A}} = e^{\boldsymbol{0}} = \boldsymbol{I}$. The second statement is (a) with $\boldsymbol{A}t$ and $\boldsymbol{A}s$, which commute.

(c) The inner factors $\boldsymbol{V}^{-1}\boldsymbol{V}$ cancel in $(\boldsymbol{V}\boldsymbol{B}\boldsymbol{V}^{-1})^k = \boldsymbol{V}\boldsymbol{B}^k\boldsymbol{V}^{-1}$, so the partial sums of $e^{\boldsymbol{V}\boldsymbol{B}\boldsymbol{V}^{-1}}$ equal $\boldsymbol{V}$ times those of $e^{\boldsymbol{B}}$ times $\boldsymbol{V}^{-1}$. Let $k \to \infty$. For a diagonal matrix, $\boldsymbol{D}^k = \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k)$, so the series acts entry by entry along the diagonal and gives e^{λ_j} there.

(d) For $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $\boldsymbol{A}^2 = \boldsymbol{0}$, so $e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, whereas the entrywise exponential is $\begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix}$. □

Corollary 6.1. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be constant. The resolvent of (27) is $\boldsymbol{R}(t,s) = e^{\boldsymbol{A}(t-s)}$, and the solution of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}(t)$, $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, with $\boldsymbol{f}$ continuous on $\mathbb{R}$, is*

$$\boxed{\boldsymbol{y}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{y}_0 + \int_{t_0}^{t} e^{\boldsymbol{A}(t-s)}\boldsymbol{f}(s)\,ds.}$$

Proof. By Theorem 6.6 and Definition 6.2, $\boldsymbol{R}(t,s) = e^{\boldsymbol{A}t}(e^{\boldsymbol{A}s})^{-1} = e^{\boldsymbol{A}t}e^{-\boldsymbol{A}s} = e^{\boldsymbol{A}(t-s)}$, by Proposition 6.5(b). The formula is Theorem 6.3. □

Computing the matrix exponential. Theorem 6.6 and Proposition 6.4(b) give a general recipe: for *any* fundamental matrix $\boldsymbol{\Phi}$ of (27),

$$e^{\boldsymbol{A}t} = \boldsymbol{R}(t,0) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}.$$

So the solutions of Section 6.3.1 give the exponential. Three cases cover every 2×2 matrix.

(1) *Diagonalizable $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$.* By Proposition 6.5(c),

$$\boxed{e^{\boldsymbol{A}t} = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1} = \boldsymbol{V}\operatorname{diag}\left(e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right)\boldsymbol{V}^{-1},}$$

the fundamental matrix of Theorem 6.5 times $\boldsymbol{V}^{-1}$. For complex eigenvalues, the identity holds in $\mathbb{C}^{n\times n}$ and the result is real.

(2) *A double eigenvalue λ_0.* Write $\boldsymbol{A} = \lambda_0\boldsymbol{I} + \boldsymbol{N}$, where $\boldsymbol{N}^2 = \boldsymbol{0}$ by the Cayley–Hamilton theorem. Since $\lambda_0\boldsymbol{I}$ and $\boldsymbol{N}$ commute, Proposition 6.5(a) gives

$$\boxed{e^{\boldsymbol{A}t} = e^{\lambda_0 t}e^{\boldsymbol{N}t} = e^{\lambda_0 t}\left(\boldsymbol{I} + t\boldsymbol{N}\right).}$$

The same argument applies whenever $\boldsymbol{N}^m = \boldsymbol{0}$ for some m ($\boldsymbol{N}$ is *nilpotent*): the series for $e^{\boldsymbol{N}t}$ stops after m terms.

(3) *A complex pair $\sigma \pm i\tau$.* The matrix $\boldsymbol{A} = \begin{pmatrix} \sigma & -\tau \\ \tau & \sigma \end{pmatrix} = \sigma\boldsymbol{I} + \tau\boldsymbol{J}$ with $\boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $\boldsymbol{J}^2 = -\boldsymbol{I}$, so the even and odd powers of $\tau t\boldsymbol{J}$ reproduce the series of cosine and sine:

$$\boxed{e^{\boldsymbol{A}t} = e^{\sigma t}e^{\tau t\boldsymbol{J}} = e^{\sigma t}\begin{pmatrix} \cos(\tau t) & -\sin(\tau t) \\ \sin(\tau t) & \cos(\tau t) \end{pmatrix},}$$

a rotation by the angle τt scaled by $e^{\sigma t}$. For a general real matrix with a complex pair, build $\boldsymbol{\Phi}$ from the real solutions $\boldsymbol{y}_1, \boldsymbol{y}_2$ of Section 6.3.1. Then $e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}$.

Example 6.8. For $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ of Example 6.5, case (1) gives

$$e^{\boldsymbol{A}t} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}.$$

As a check, $e^{\boldsymbol{A}\cdot 0} = \boldsymbol{I}$, and $e^{\boldsymbol{A}t}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^{2t} + 2e^{3t} \\ e^{2t} + 2e^{3t} \end{pmatrix}$ agrees with Example 6.5.

Example 6.9. Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$. Then $\operatorname{tr}\boldsymbol{A} = 2$ and $\det\boldsymbol{A} = 1$, so $p(\lambda) = (\lambda - 1)^2$ and $\lambda_0 = 1$ is a double eigenvalue. The eigenvectors solve $2v_1 + v_2 = 0$, so the eigenspace is a line and $\boldsymbol{A}$ is not diagonalizable. Case (2) applies with

$$\boldsymbol{N} = \boldsymbol{A} - \boldsymbol{I} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}, \qquad \boldsymbol{N}^2 = \begin{pmatrix} 4-4 & 2-2 \\ -8+8 & -4+4 \end{pmatrix} = \boldsymbol{0},$$

so that

$$e^{\boldsymbol{A}t} = e^t \begin{pmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{pmatrix}.$$

The solution with $\boldsymbol{y}(0) = (1, 0)^\top$ is the first column, $\boldsymbol{y}(t) = e^t(1+2t, -4t)^\top$, and the resolvent is $\boldsymbol{R}(t, s) = e^{\boldsymbol{A}(t-s)}$.

6.3.3 Solution by the Laplace transform

The Laplace transform of Chapter 4 applies componentwise to systems with constant coefficients. Consider $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}(t)$ with $\boldsymbol{y}(0) = \boldsymbol{y}_0$, where the components of $\boldsymbol{f}$ are piecewise continuous and of exponential order, and let $\boldsymbol{Y}(s) = (Y_1(s), \dots, Y_n(s))^\top$ and $\boldsymbol{F}(s)$ collect the transforms $y_j \circ\!-\!\bullet\, Y_j$ and $f_j \circ\!-\!\bullet\, F_j$. Rule f) of Theorem 4.2, applied to each component, gives $s\boldsymbol{Y}(s) - \boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{Y}(s) + \boldsymbol{F}(s)$, that is,

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{Y}(s) = \boldsymbol{y}_0 + \boldsymbol{F}(s) \quad \Longrightarrow \quad \boxed{\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\left(\boldsymbol{y}_0 + \boldsymbol{F}(s)\right),}$$

valid for every s larger than the real parts of all eigenvalues of $\boldsymbol{A}$ at which $\boldsymbol{F}(s)$ is defined, since such an s is not an eigenvalue and $s\boldsymbol{I} - \boldsymbol{A}$ is invertible. Procedure:

(1) transform each equation of the system;

(2) solve the resulting $n \times n$ linear system for $Y_1(s), \dots, Y_n(s)$;

(3) invert each $Y_j(s)$ with the table of Section 4.2.

Comparing with $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$ in the homogeneous case, we find

$$\mathcal{L}\left[e^{\boldsymbol{A}t}\right](s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} \quad \text{(entrywise)}.$$

In linear algebra, $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ is called the resolvent of $\boldsymbol{A}$, which explains the name of $\boldsymbol{R}(t, s)$ in Definition 6.2. As Remark 6.1 predicts, for the companion matrix $\boldsymbol{C}$ of an nth-order equation, expanding the determinant shows that the $(1, n)$ entry of $(s\boldsymbol{I} - \boldsymbol{C})^{-1}$ is $1/p(s)$, the transfer function of Section 4.6. For the harmonic oscillator $y_1' = y_2$, $y_2' = -\omega_0^2 y_1$, for instance,

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{s^2 + \omega_0^2}\begin{pmatrix} s & 1 \\ -\omega_0^2 & s \end{pmatrix} \quad \bullet\!-\!\circ \quad e^{\boldsymbol{A}t} = \begin{pmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix},$$

with the Green's function $\sin(\omega_0 t)/\omega_0$ of Example 4.10 in the $(1, 2)$ entry.

6.4 Phase portraits and stability in the plane

Section 2.4 analyzed scalar autonomous equations $\dot{x} = f(x)$ by drawing the vector field on a line. We now do the same in the plane for $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ with constant $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$, and ask the same questions: What are the equilibria? How do solutions behave near them? Do solutions tend to an equilibrium as $t \to \infty$?

Definition 6.4. A constant solution $\boldsymbol{y}(t) \equiv \boldsymbol{y}^*$ of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ is called an *equilibrium solution*, and the point $\boldsymbol{y}^* \in \mathbb{R}^n$ is called an *equilibrium* (or *fixed point*, or *critical point*). Equivalently, $\boldsymbol{A}\boldsymbol{y}^* = \boldsymbol{0}$.

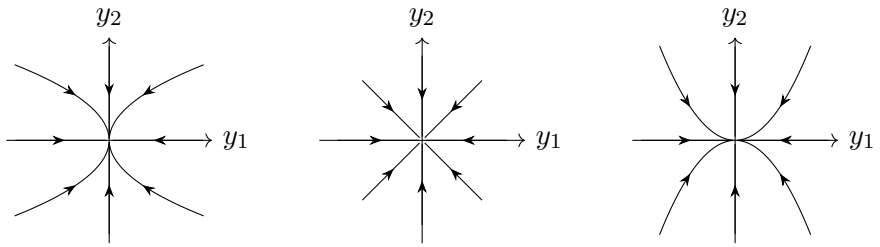
If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{y}^* = \boldsymbol{0}$ is the only equilibrium. If $\det \boldsymbol{A} = 0$, the equilibria form the kernel of $\boldsymbol{A}$, a line or the whole plane.

A solution $t \mapsto \boldsymbol{y}(t)$ traces a curve in the *phase plane* (y_1, y_2), called a *trajectory* (or *orbit*) and marked with an arrow in the direction of increasing t. A *phase portrait* shows representative trajectories. Two facts hold for every phase portrait. First, since $\boldsymbol{A}$ does not depend on t, the time shift $t \mapsto \boldsymbol{y}(t + c)$ of a solution is a solution, so a trajectory does not depend on when the solution starts on it. Second, distinct trajectories never cross. By uniqueness (Theorem 6.1), two trajectories through one point come from solutions that coincide after a time shift.

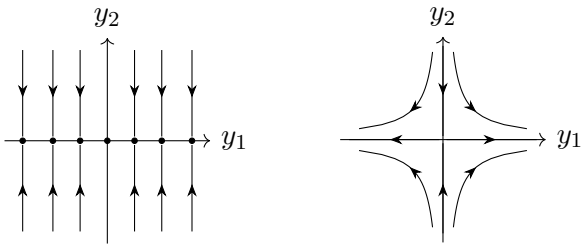
Example 6.10. Consider

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \qquad \boldsymbol{y}(0) = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}.$$

The system is decoupled, and the solution is $y_1(t) = y_{01}e^{at}$, $y_2(t) = y_{02}e^{-t}$. Let's draw the phase portrait for different values of a.



When $a < 0$, both components decay, so $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$ for every initial value. The direction of approach depends on which component decays faster. For $a < -1$, y_1 decays faster and the trajectories become tangent to the y_2-axis. For $-1 < a < 0$, they become tangent to the y_1-axis, and for $a = -1$, they are rays. (Eliminating t gives the curves $|y_2| = C|y_1|^{-1/a}$.)



If $a = 0$, then $y_1(t) = y_{01}$ is constant, the y_1-axis is a *line of equilibria*, and every trajectory approaches it vertically. If $a > 0$, then $|y_1(t)| \to \infty$ for $y_{01} \neq 0$ while $y_2(t) \to 0$, so trajectories leave the origin, except those on the y_2-axis, which converge to it. For $a < 0$ with $a \neq -1$, the origin is a *stable node*, for $a = -1$ a *stable star*, and for $a > 0$ a *saddle point*.

6.4.1 Classification of equilibria

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det \boldsymbol{A} \neq 0$. By Section 6.1, the eigenvalues are the roots of $\lambda^2 - (\operatorname{tr} \boldsymbol{A})\lambda + \det \boldsymbol{A}$, namely

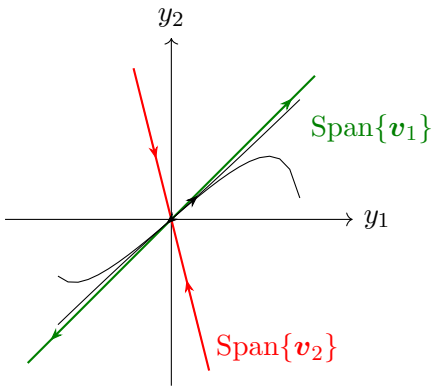
$$\lambda_{1,2} = \frac{\operatorname{tr} \boldsymbol{A} \pm \sqrt{(\operatorname{tr} \boldsymbol{A})^2 - 4 \det \boldsymbol{A}}}{2}, \tag{28}$$

so the trace and the determinant determine the eigenvalues. For $\lambda_1 \neq \lambda_2$, Section 6.3.1 gives every solution as

$$\boldsymbol{y}(t) = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_2 t} \boldsymbol{v}_2,$$

with real or conjugate-complex constants and eigenvectors, and for $\lambda_1 = \lambda_2$ the repeated-eigenvalue case applies. The eigenvalues and eigenvectors therefore determine the long-term behavior, and the portraits of Example 6.10 are typical, with the lines $\operatorname{Span}\{\boldsymbol{v}_1\}$ and $\operatorname{Span}\{\boldsymbol{v}_2\}$ in place of the axes.

Example 6.11. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, so that $\operatorname{tr} \boldsymbol{A} = -1$ and $\det \boldsymbol{A} = -6$. Then $(\operatorname{tr} \boldsymbol{A})^2 - 4 \det \boldsymbol{A} = 25$, and (28) gives $\lambda_1 = 2$ and $\lambda_2 = -3$, with eigenvectors $\boldsymbol{v}_1 = (1, 1)^\top$ and $\boldsymbol{v}_2 = (1, -4)^\top$. The solution $c_1 e^{2t} \boldsymbol{v}_1$ grows and the solution $c_2 e^{-3t} \boldsymbol{v}_2$ decays, so the origin is a saddle point:



The line Span$\{\boldsymbol{v}_2\}$, along which trajectories approach the origin, is the *stable manifold* of the equilibrium, and Span$\{\boldsymbol{v}_1\}$, along which they leave it, the *unstable manifold*. Every other trajectory comes in along the stable manifold and leaves along the unstable manifold.

Definition 6.5. Let $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with $\det \boldsymbol{A} \neq 0$ have eigenvalues λ_1, λ_2, and let $\boldsymbol{y}^* = \boldsymbol{0}$ be the equilibrium of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$.

i) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2 < 0$, then $\boldsymbol{y}^*$ is a *stable node*.

ii) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 > \lambda_2 > 0$, then $\boldsymbol{y}^*$ is an *unstable node*.

iii) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_2 < 0 < \lambda_1$, then $\boldsymbol{y}^*$ is a *saddle point*.

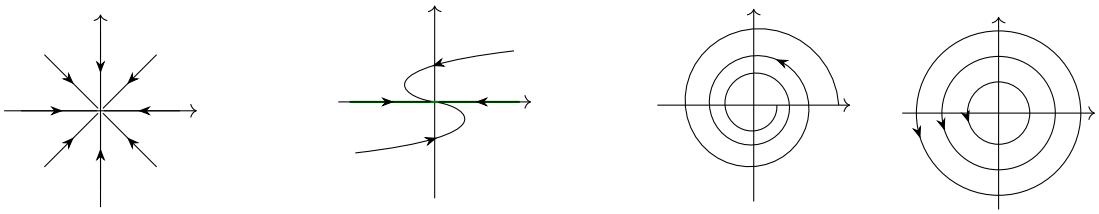
iv) If $\lambda_1 = \lambda_2 \in \mathbb{R}$ and $\boldsymbol{A} = \lambda_1 \boldsymbol{I}$, then $\boldsymbol{y}^*$ is a stable ($\lambda_1 < 0$) or unstable ($\lambda_1 > 0$) *star node*.

v) If $\lambda_1 = \lambda_2 \in \mathbb{R}$ and $\boldsymbol{A} \neq \lambda_1 \boldsymbol{I}$ (only one eigenvector direction), then $\boldsymbol{y}^*$ is a stable ($\lambda_1 < 0$) or unstable ($\lambda_1 > 0$) *degenerate node*.

vi) If $\lambda_{1,2} = \sigma \pm i\tau$ with $\tau \neq 0$ and $\sigma \neq 0$, then $\boldsymbol{y}^*$ is a stable ($\sigma < 0$) or unstable ($\sigma > 0$) *spiral*. (Note that $2\sigma = \lambda_1 + \lambda_2 = \operatorname{tr} \boldsymbol{A}$.)

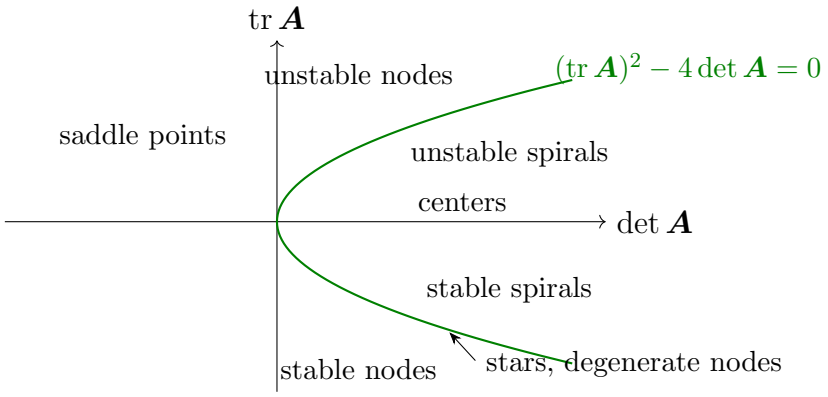
vii) If $\lambda_{1,2} = \pm i\tau$ with $\tau \neq 0$, then $\boldsymbol{y}^*$ is a *center*. The trajectories are concentric ellipses (circles only for special $\boldsymbol{A}$).

Cases i)–iii) look like the node and saddle portraits above, with Span$\{\boldsymbol{v}_1\}$ and Span$\{\boldsymbol{v}_2\}$ in place of the axes. Portraits for iv)–vii) follow.



stable star node | degenerate node ($\lambda_1 < 0$) | stable spiral | center

Since (28) expresses the eigenvalues through $\operatorname{tr} \boldsymbol{A}$ and $\det \boldsymbol{A}$, one diagram summarizes every type. The parabola $(\operatorname{tr} \boldsymbol{A})^2 = 4 \det \boldsymbol{A}$ separates real from complex eigenvalues. A negative determinant gives real eigenvalues of opposite signs (saddles). For $\det \boldsymbol{A} > 0$, the sign of $\operatorname{tr} \boldsymbol{A} = \lambda_1 + \lambda_2$ decides between stable and unstable.



Example 6.12 (The damped oscillator in the phase plane). Consider the damped oscillator of Section 3.4, $my'' + \gamma y' + ky = 0$ with $m, k > 0$ and $\gamma \geq 0$. For $\boldsymbol{y} = (y, y')^\top$, it becomes the planar system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ with

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix}, \qquad \operatorname{tr} \boldsymbol{A} = -\frac{\gamma}{m}, \qquad \det \boldsymbol{A} = \frac{k}{m} > 0.$$

The point $(\det \boldsymbol{A}, \operatorname{tr} \boldsymbol{A})$ lies in the right half of the diagram, on or below the horizontal axis, and $(\operatorname{tr} \boldsymbol{A})^2 - 4 \det \boldsymbol{A} = (\gamma^2 - 4km)/m^2$. The three regimes of Section 3.4 are three parts of the diagram: overdamped ($\gamma^2 > 4km$), a stable node; critically damped ($\gamma^2 = 4km$), a stable degenerate node, since $\boldsymbol{A} \neq \lambda \boldsymbol{I}$; underdamped ($\gamma^2 < 4km$), a stable spiral, whose trajectories wind around the origin as the mass oscillates with decaying amplitude. Without damping ($\gamma = 0$), the origin is a center, and the closed trajectories are the ellipses $\frac{k}{2}y^2 + \frac{m}{2}(y')^2 = \text{const}$ of constant energy.

6.4.2 Stability

The words "stable" and "unstable" in Definition 6.5 have a precise meaning, which we state for $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ with $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Recall that every solution exists for all $t \in \mathbb{R}$ and equals $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$ (Corollary 6.1).

Definition 6.6. The equilibrium $\boldsymbol{y}^* = \boldsymbol{0}$ of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ is

a) *Lyapunov stable* (or simply *stable*) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\boldsymbol{y}(0)\| < \delta$ implies $\|\boldsymbol{y}(t)\| < \varepsilon$ for every $t \geq 0$;

b) *asymptotically stable* if it is Lyapunov stable and $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$ for every solution $\boldsymbol{y}$;

c) *unstable* if it is not Lyapunov stable.

A solution that starts close to a stable equilibrium stays close for all later times. An asymptotically stable equilibrium also attracts every solution. For linear systems, stability is a property of $e^{\boldsymbol{A}t}$.

Lemma 6.1. *The equilibrium* $\boldsymbol{0}$ *of* $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ *is Lyapunov stable if and only if* $\sup_{t\geq 0} \left\|e^{\boldsymbol{A}t}\right\| < \infty$*, and asymptotically stable if and only if* $e^{\boldsymbol{A}t} \to \boldsymbol{0}$ *as* $t \to \infty$*.*

Proof. If $M := \sup_{t\geq 0} \left\|e^{\boldsymbol{A}t}\right\| < \infty$, then $\delta = \varepsilon/M$ works: $\|\boldsymbol{y}(0)\| < \delta$ gives $\|\boldsymbol{y}(t)\| = \left\|e^{\boldsymbol{A}t}\boldsymbol{y}(0)\right\| \leq M\,\|\boldsymbol{y}(0)\| < \varepsilon$ for every $t \geq 0$. Conversely, let $\boldsymbol{0}$ be Lyapunov stable and let δ correspond to $\varepsilon = 1$. For every unit vector $\boldsymbol{u}$ and $t \geq 0$, the solution with $\boldsymbol{y}(0) = \frac{\delta}{2}\boldsymbol{u}$ satisfies $\left\|e^{\boldsymbol{A}t}\frac{\delta}{2}\boldsymbol{u}\right\| < 1$, so $\left\|e^{\boldsymbol{A}t}\boldsymbol{u}\right\| < 2/\delta$, and the maximum over $\boldsymbol{u}$ gives $\left\|e^{\boldsymbol{A}t}\right\| \leq 2/\delta$.

If $e^{\boldsymbol{A}t} \to \boldsymbol{0}$, then $t \mapsto \left\|e^{\boldsymbol{A}t}\right\|$ is continuous on $[0,\infty)$ with limit 0, hence bounded, so $\boldsymbol{0}$ is Lyapunov stable, and $\|\boldsymbol{y}(t)\| \leq \left\|e^{\boldsymbol{A}t}\right\|\,\|\boldsymbol{y}(0)\| \to 0$ for every solution. Conversely, if every solution tends to $\boldsymbol{0}$, so do the columns of $e^{\boldsymbol{A}t}$, the solutions with initial values $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. □

Theorem 6.7 (Stability of planar linear systems)**.** *Let* $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ *have eigenvalues* λ_1, λ_2*. The equilibrium* $\boldsymbol{0}$ *of* $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ *is*

(a) asymptotically stable if and only if $\operatorname{Re}\lambda_1 < 0$ *and* $\operatorname{Re}\lambda_2 < 0$*;*

(b) Lyapunov stable if and only if $\operatorname{Re}\lambda_1 \leq 0$ *and* $\operatorname{Re}\lambda_2 \leq 0$*, and, in case* $\lambda_1 = \lambda_2 = 0$*,* $\boldsymbol{A} = \boldsymbol{0}$*;*

(c) unstable in all other cases: if some eigenvalue has positive real part, or if $\lambda_1 = \lambda_2 = 0$ *and* $\boldsymbol{A} \neq \boldsymbol{0}$*.*

Proof. By Lemma 6.1, we must decide when $e^{\boldsymbol{A}t}$ stays bounded, and when it tends to $\boldsymbol{0}$, as $t \to \infty$. The three cases of Section 6.3.2 give $e^{\boldsymbol{A}t}$ explicitly.

Distinct real eigenvalues. Then $e^{\boldsymbol{A}t} = \boldsymbol{V}\operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t})\boldsymbol{V}^{-1}$, so every entry is a constant-coefficient combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. If $\lambda_1, \lambda_2 \leq 0$, all entries are bounded, and if $\lambda_1, \lambda_2 < 0$, they tend to 0. If $\lambda_1 > 0$, then $e^{\boldsymbol{A}t}\boldsymbol{v}_1 = e^{\lambda_1 t}\boldsymbol{v}_1$ is unbounded. If $\lambda_1 = 0 > \lambda_2$, then $e^{\boldsymbol{A}t}\boldsymbol{v}_1 = \boldsymbol{v}_1$ does not tend to $\boldsymbol{0}$, so the equilibrium is stable but not asymptotically stable.

A complex pair $\sigma \pm i\tau$*.* Then $e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}$ with the real fundamental matrix built from $\boldsymbol{y}_1, \boldsymbol{y}_2$ of Section 6.3.1, so every entry has the form $e^{\sigma t}\,(\alpha\cos(\tau t) + \beta\sin(\tau t))$: bounded if $\sigma \leq 0$, tending to 0 if $\sigma < 0$. For $\sigma > 0$, $\left\|e^{\boldsymbol{A}t}\boldsymbol{y}_1(0)\right\| = \|\boldsymbol{y}_1(t)\| = e^{\sigma t}\,\|\boldsymbol{a}\cos(\tau t) - \boldsymbol{b}\sin(\tau t)\|$, and the last norm is continuous, periodic, and never zero, since $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. It is bounded below by a positive constant, so $e^{\boldsymbol{A}t}$ is unbounded. For $\sigma = 0$, the entries do not tend to 0.

A double eigenvalue λ_0*.* Then $e^{\boldsymbol{A}t} = e^{\lambda_0 t}(\boldsymbol{I} + t\boldsymbol{N})$ with $\boldsymbol{N} = \boldsymbol{A} - \lambda_0\boldsymbol{I}$. If $\lambda_0 < 0$, both $e^{\lambda_0 t}$ and $te^{\lambda_0 t}$ tend to 0. If $\lambda_0 > 0$, then $e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda_0 t}\boldsymbol{v}$ is unbounded for an eigenvector $\boldsymbol{v}$. If

$\lambda_0 = 0$, then $e^{\boldsymbol{A}t} = \boldsymbol{I} + t\boldsymbol{N}$ is bounded if and only if $\boldsymbol{N} = \boldsymbol{0}$, that is, $\boldsymbol{A} = \boldsymbol{0}$, and never tends to $\boldsymbol{0}$.

Collecting the cases proves (a), (b), and (c). □

Remark 6.4. Theorem 6.7 holds for every n with one change in (b): every eigenvalue with $\operatorname{Re}\lambda = 0$ must be *semisimple*, that is, its eigenspace has dimension equal to its multiplicity as a root of p, so that no chain of generalized eigenvectors, and no factor t, appears in Remark 6.2. The proof is the same, with the Jordan normal form supplying $e^{\boldsymbol{A}t}$. In the plane, the trace–determinant diagram shows the sign of $\operatorname{Re}\lambda$. The equilibrium is asymptotically stable exactly when $\det \boldsymbol{A} > 0$ and $\operatorname{tr} \boldsymbol{A} < 0$. Chapter 7 uses this theorem to study equilibria of nonlinear systems by linearization.

Example 6.13 (Love affairs — Strogatz, 1988). Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let $R(t)$ be Romeo's love/hate for Juliet at time t, and let $J(t)$ be Juliet's love/hate for Romeo at time t. Positive values of R, J signify love; negative values signify hate. Then a model for their star-crossed romance is

$$\begin{cases} R'(t) = \mu J(t), & \mu > 0, \\ J'(t) = -\nu R(t), & \nu > 0. \end{cases}$$

So $\boldsymbol{A} = \begin{pmatrix} 0 & \mu \\ -\nu & 0 \end{pmatrix}$, with $\operatorname{tr} \boldsymbol{A} = 0$ and $\det \boldsymbol{A} = \mu\nu > 0$, and $\lambda_{1,2} = \pm i\sqrt{\mu\nu}$. Hence the origin is a center: the trajectories are the ellipses $\nu R^2 + \mu J^2 = \text{const}$ (differentiate the left-hand side along a solution to see that it is constant), traversed clockwise.

Outcome: a never-ending cycle of love and hate. The equilibrium is stable, but not asymptotically stable: the lovers never settle down.

In general, we would have

$$\begin{pmatrix} R'(t) \\ J'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} R(t) \\ J(t) \end{pmatrix}.$$

The signs of a, b, c, and d then determine the lovers' "romantic types". For Romeo,

1) $a > 0$, $b > 0$: "Eager Beaver";

2) $a > 0$, $b < 0$: "Narcissistic Nerd";

3) $a < 0$, $b > 0$: "Bashful Budder";

4) $a < 0$, $b < 0$: "Malevolent Misanthrope".

Can a "narcissistic nerd" find love with a "bashful budder"? Classify the equilibrium of the resulting system with the trace–determinant diagram to find out!

Summary.

1) A linear system $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y} + \boldsymbol{f}(t)$ with continuous coefficients on an open interval I has exactly one solution on all of I for each initial value $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ (proof in Chapter 7). Every nth-order linear equation is such a system, via its companion matrix.

2) The homogeneous solutions form a vector space of dimension n. Any n solutions are linearly independent if and only if their Wronskian $W = \det \boldsymbol{\Phi}$ is nonzero at one point, if and only if it is nonzero everywhere, and $W(t) = W(t_0) \exp \int_{t_0}^{t} \operatorname{tr} \boldsymbol{A}$ (Abel–Liouville). Through the companion matrix, this gives the nth-order structure theorem, Wronskian test, and Abel's identity.

3) The resolvent $\boldsymbol{R}(t, s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}$ maps the state at time s to the state at time t, and Duhamel's formula $\boldsymbol{y}(t) = \boldsymbol{R}(t, t_0)\boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{R}(t, s)\boldsymbol{f}(s)\, ds$ solves the inhomogeneous problem. The Green's function of Chapter 4 is an entry of the resolvent.

4) For constant $\boldsymbol{A}$, the Ansatz $e^{\lambda t}\boldsymbol{v}$ works when $\boldsymbol{v}$ is an eigenvector. Complex pairs give real solutions through real and imaginary parts, and a double eigenvalue with a single eigenvector direction requires a generalized eigenvector and a factor t.

5) The matrix exponential $e^{\boldsymbol{A}t} = \sum_k t^k \boldsymbol{A}^k / k!$ is the fundamental matrix with $e^{\boldsymbol{A}\cdot 0} = \boldsymbol{I}$, the resolvent is $e^{\boldsymbol{A}(t-s)}$, and $e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}$ for any fundamental matrix. The Laplace transform gives $\mathcal{L}[e^{\boldsymbol{A}t}](s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$.

6) In the plane, the trace and determinant of $\boldsymbol{A}$ classify the equilibrium at the origin (node, saddle, star, degenerate node, spiral, center). The origin is asymptotically stable if and only if every eigenvalue has negative real part, and Lyapunov stable if and only if no eigenvalue has positive real part and a double zero eigenvalue occurs only for $\boldsymbol{A} = \boldsymbol{0}$.