

5 Series solutions of second-order ODEs

The characteristic polynomial of Section 3.2 applies only to constant coefficients. Many equations in practice have variable coefficients.

- 1) The Bessel equation $t^2 y''(t) + ty'(t) + (t^2 - n^2)y(t) = 0$, where n is a nonnegative integer, arises in wave phenomena (fluid dynamics, optics, electromagnetism).
- 2) The associated Laguerre equation $ty''(t) + (\mu + 1 - t)y'(t) + ny(t) = 0$, where μ is real and n is a nonnegative integer. Its solutions give the radial part of the wave function of the hydrogen atom.
- 3) The Airy equation $y''(t) - ty(t) = 0$ arises in quantum mechanics, astronomy, optics, ...

We represent their solutions by power series. Consider the ODE

$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0, \quad (16)$$

where P , Q , and R are given functions. On an open interval where P does not vanish and Q/P and R/P are continuous, dividing (16) by P gives (8) with $a_1 = Q/P$ and $a_0 = R/P$, and the theory of Section 3.1 applies. We expand about a point t_0 with $P(t_0) \neq 0$ in Section 5.1 and about certain points with $P(t_0) = 0$ in Section 5.2.

5.1 Series solutions around an ordinary point

We write the solution of (16) as a Taylor series with coefficients $\{c_k\}_{k=0}^\infty$ about the point t_0 :

$$y(t) = \sum_{k=0}^{\infty} c_k (t - t_0)^k.$$

Facts about power series.

- i) (Radius of convergence) Every power series $\sum_{k=0}^{\infty} c_k (t - t_0)^k$ has a *radius of convergence* $\rho \in [0, \infty]$: the series converges absolutely for $|t - t_0| < \rho$ and diverges for $|t - t_0| > \rho$. The Cauchy–Hadamard formula gives $\rho = 1/\limsup_{k \rightarrow \infty} |c_k|^{1/k}$, with $1/0 = \infty$ and $1/\infty = 0$. If $c_k \neq 0$ for all large k and $\lim_{k \rightarrow \infty} |c_k/c_{k+1}|$ exists in $[0, \infty]$, then this limit equals ρ (ratio test). If infinitely many coefficients vanish, as in a series of even powers, apply the ratio test to the nonzero terms; the test gives ρ when their ratios have a limit.
- ii) (Term-by-term differentiation) On $|t - t_0| < \rho$, the function $y(t) = \sum_{k=0}^{\infty} c_k (t - t_0)^k$ has derivatives of all orders, obtained by differentiating term by term:

$$y'(t) = \sum_{k=1}^{\infty} k c_k (t - t_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} (t - t_0)^k.$$

The differentiated series has the same radius of convergence. In particular, $c_k = y^{(k)}(t_0)/k!$ for every $k \geq 0$.

iii) (Identity theorem) If $\sum_{k=0}^{\infty} c_k(t-t_0)^k = \sum_{k=0}^{\infty} d_k(t-t_0)^k$ for all t in an open interval containing t_0 , then $c_k = d_k$ for every $k \geq 0$, because both sides have the same derivatives at t_0 .

iv) (Products) If two power series converge for $|t-t_0| < \rho$, then so does their product, and

$$\left(\sum_{k=0}^{\infty} c_k(t-t_0)^k \right) \left(\sum_{k=0}^{\infty} d_k(t-t_0)^k \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k c_{k-j}d_j \right) (t-t_0)^k.$$

A function f is *analytic* at t_0 if it equals a power series with positive radius of convergence on an open interval containing t_0 . Polynomials, e^t , $\cos t$, and $\sin t$ are analytic at every point, and their Taylor series have infinite radius of convergence. Sums and products of functions analytic at t_0 are analytic at t_0 , and so is a quotient f/g with $g(t_0) \neq 0$. If f and g are polynomials, complex analysis bounds the radius of convergence from below: the Taylor series of f/g about t_0 converges at least for $|t-t_0| < d$, where d is the distance from t_0 to the nearest complex zero of g . For example, $1/(1+t^2) = \sum_{k=0}^{\infty} (-1)^k t^{2k}$ converges only for $|t| < 1$, and the zeros $\pm i$ of $1+t^2$ lie at distance 1 from 0.

Definition 5.1. Let P , Q , and R be analytic at t_0 . The point t_0 is an *ordinary point* of (16) if $P(t_0) \neq 0$, and a *singular point* of (16) if $P(t_0) = 0$.

At an ordinary point, the coefficients $a_1 = Q/P$ and $a_0 = R/P$ of the normal form are analytic at t_0 .

Theorem 5.1 (Series solutions at an ordinary point). *Let $t_0 \in \mathbb{R}$ and $\rho \in (0, \infty]$, and suppose that a_1 and a_0 are given by power series about t_0 that converge for $|t-t_0| < \rho$. Then, for all $c_0, c_1 \in \mathbb{R}$, the initial value problem*

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0, \quad y(t_0) = c_0, \quad y'(t_0) = c_1,$$

has a unique solution on $I = (t_0 - \rho, t_0 + \rho)$. This solution is a power series $y(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k$ that converges on I , and the coefficients c_k with $k \geq 2$ are determined by c_0 and c_1 through the recurrence obtained by substituting the series into the ODE.

Proof sketch. Shifting the variable, we may assume that $t_0 = 0$. Write $a_1(t) = \sum_{k=0}^{\infty} a_{1,k}t^k$ and $a_0(t) = \sum_{k=0}^{\infty} a_{0,k}t^k$. Substituting $y(t) = \sum_{k=0}^{\infty} c_k t^k$ into the ODE and using facts ii)–iv), we find that y solves the ODE if and only if

$$(k+2)(k+1)c_{k+2} = - \sum_{j=0}^k ((j+1)a_{1,k-j}c_{j+1} + a_{0,k-j}c_j) \quad \text{for every } k \geq 0.$$

The right-hand side involves only $c_0, \dots, c_{k+1}$, so c_0 and c_1 determine every coefficient. To prove convergence, fix $s \in (0, \rho)$. The series of a_1 and a_0 converge at $t = s$, so their terms are bounded: there exists $M > 0$ such that $|a_{1,k}| \leq Ms^{-k}$ and $|a_{0,k}| \leq Ms^{-k}$ for every $k \geq 0$. Replacing each term of the recurrence by this bound defines a *majorant* sequence: $C_0 = |c_0|$, $C_1 = |c_1|$, and $(k+2)(k+1)C_{k+2} = M \sum_{j=0}^k s^{j-k}((j+1)C_{j+1} + C_j)$ for $k \geq 0$. Induction on k gives $|c_k| \leq C_k$. If $(c_0, c_1) \neq (0, 0)$, the recurrence for C_k gives $C_{k+1}/C_k \rightarrow 1/s$. By the ratio test and comparison, $\sum_k c_k t^k$ converges for $|t| < s$. Since $s \in (0, \rho)$ was arbitrary, the series converges on I , and by construction it solves the initial value problem there. Uniqueness follows from Theorem 3.1, because a_1 and a_0 are continuous on I . $\square$

Corollary 5.1. *Under the hypotheses of Theorem 5.1, let y_1 and y_2 be the series solutions with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, respectively. Then $\{y_1, y_2\}$ is a fundamental set of solutions on I , and every solution on I is a power series about t_0 that converges on I .*

Proof. The Wronskian at t_0 equals $W(t_0) = 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$, so y_1 and y_2 are linearly independent by Corollary 3.1. By Theorem 3.2, every solution y on I equals $y(t_0)y_1 + y'(t_0)y_2$, a power series that converges on I . $\square$

If P , Q , and R are polynomials and t_0 is an ordinary point, then by the bound from complex analysis, Theorem 5.1 applies with ρ equal to the distance from t_0 to the nearest complex zero of P ($\rho = \infty$ if P has no zeros).

Procedure:

- 1) Write the solution in series form.
- 2) Plug into the ODE.
- 3) Order the terms and find recurrence relations.

Example 5.1 (The Airy equation). Let's find the series solutions of the Airy equation $y''(t) - ty(t) = 0$ about $t_0 = 1$. The coefficients $a_1 = 0$ and $a_0(t) = -t$ are polynomials, so Theorem 5.1 applies with $\rho = \infty$. We write $y(t) = \sum_{k=0}^{\infty} c_k(t-1)^k$. Differentiating twice term by term and shifting the index gives

$$y''(t) = \sum_{k=2}^{\infty} k(k-1)c_k(t-1)^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}(t-1)^k.$$

To expand the term $ty(t)$ about $t_0 = 1$, write $t = (t-1) + 1$:

$$ty(t) = \sum_{k=0}^{\infty} c_k(t-1)^{k+1} + \sum_{k=0}^{\infty} c_k(t-1)^k = \sum_{k=1}^{\infty} c_{k-1}(t-1)^k + \sum_{k=0}^{\infty} c_k(t-1)^k.$$

Plugging both expansions into the ODE gives

$$0 = y''(t) - ty(t) = (2c_2 - c_0) + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_k - c_{k-1}](t-1)^k.$$

By the identity theorem, every coefficient vanishes: $2c_2 = c_0$, and $(k+2)(k+1)c_{k+2} = c_k + c_{k-1}$ for all $k \geq 1$. The coefficients c_0 and c_1 are free, $c_2 = c_0/2$, and the cases $k = 1$ and $k = 2$ give $c_3 = (c_0 + c_1)/6$ and $c_4 = (c_1 + c_2)/12 = c_1/12 + c_0/24$. Hence

$$\begin{aligned} y(t) = c_0 & \left(1 + \frac{(t-1)^2}{2} + \frac{(t-1)^3}{6} + \frac{(t-1)^4}{24} + \dots \right) \\ & + c_1 \left((t-1) + \frac{(t-1)^3}{6} + \frac{(t-1)^4}{12} + \dots \right). \end{aligned}$$

By Corollary 5.1, the two series in parentheses converge for every $t \in \mathbb{R}$ and form a fundamental set of solutions on $\mathbb{R}$.

5.2 Series solutions around a regular singular point

5.2.1 The Euler equation

Consider the *Euler equation*

$$\alpha_0 t^2 y''(t) + \beta_0 t y'(t) + \gamma_0 y(t) = 0, \quad t > 0, \quad (17)$$

where $\alpha_0 \neq 0$, β_0 , and γ_0 are real constants. The point $t_0 = 0$ is singular, and solutions need not be power series near it: in Example 3.5, the equation $2t^2 y''(t) + 3t y'(t) - y(t) = 0$ has the fundamental set $\{\sqrt{t}, 1/t\}$, and neither function is analytic at 0. Let's try the Ansatz $y(t) = t^r$ instead. Write $\text{ODE}(y)$ for the left-hand side of (17). Since $t^r = e^{r \ln t}$, we have $\frac{d}{dt} t^r = r t^{r-1}$ for every $r \in \mathbb{C}$, so

$$\text{ODE}(t^r) = \chi(r) t^r, \quad \text{where } \chi(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Hence t^r solves (17) if and only if r solves the *indicial equation* $\chi(r) = 0$. For the example, $\chi(r) = 2r(r-1) + 3r - 1 = (2r-1)(r+1)$, with roots $\frac{1}{2}$ and -1 . As in Section 3.2, three cases arise.

- (1) Distinct real roots $r_1 \neq r_2$. The general solution is $y(t) = c_1 t^{r_1} + c_2 t^{r_2}$, and the Wronskian is $(r_2 - r_1) t^{r_1+r_2-1} \neq 0$.
- (2) Complex roots $\sigma \pm i\tau$ with $\tau \neq 0$. Euler's formula gives $t^{\sigma+i\tau} = t^\sigma e^{i\tau \ln t} = t^\sigma \cos(\tau \ln t) + i t^\sigma \sin(\tau \ln t)$. The coefficients of (17) are real, so the real and imaginary parts $t^\sigma \cos(\tau \ln t)$ and $t^\sigma \sin(\tau \ln t)$ are real solutions, as in Case 2 of Section 3.2. Their Wronskian is $\tau t^{2\sigma-1} \neq 0$, and the general solution is

$$y(t) = t^\sigma (c_1 \cos(\tau \ln t) + c_2 \sin(\tau \ln t)).$$

- (3) A double root r_0 , so that $\chi(r) = \alpha_0 (r-r_0)^2$. The function $(t, r) \mapsto t^r$ is smooth for $t > 0$, and $\frac{\partial}{\partial r} t^r = t^r \ln t$. Differentiating the identity $\text{ODE}(t^r) = \chi(r) t^r$ in r and interchanging $\frac{\partial}{\partial r}$ with $\frac{d}{dt}$ by Schwarz's theorem gives

$$\text{ODE}(t^r \ln t) = \chi'(r) t^r + \chi(r) t^r \ln t.$$

At $r = r_0$, both $\chi(r_0)$ and $\chi'(r_0)$ vanish, so $t^{r_0} \ln t$ is a second solution. The Wronskian of t^{r_0} and $t^{r_0} \ln t$ is $t^{2r_0-1} \neq 0$, and the general solution is $y(t) = (c_1 + c_2 \ln t) t^{r_0}$.

The substitution $t \mapsto -t$ leaves (17) unchanged, so on $t < 0$ the same formulas hold with t replaced by $|t|$.

5.2.2 Regular singular points and the theorem of Frobenius

Definition 5.2. Let P , Q , and R be analytic at t_0 , and let t_0 be a singular point of (16). The point t_0 is a *regular singular point* if there exist functions β and γ , analytic at t_0 , such that

$$(t - t_0) \frac{Q(t)}{P(t)} = \beta(t) \quad \text{and} \quad (t - t_0)^2 \frac{R(t)}{P(t)} = \gamma(t)$$

for every $t \neq t_0$ near t_0 . Otherwise, t_0 is an *irregular singular point*.

Shifting the variable, we take $t_0 = 0$. Dividing (16) by P and multiplying by t^2 gives

$$t^2 y''(t) + t\beta(t)y'(t) + \gamma(t)y(t) = 0. \quad (18)$$

The Euler equation (17) is the case of constants $\beta = \beta_0/\alpha_0$ and $\gamma = \gamma_0/\alpha_0$, and the Bessel equation is the case $\beta(t) = 1$ and $\gamma(t) = t^2 - n^2$. Near $t = 0$, equation (18) resembles the Euler equation with $\beta \equiv \beta(0)$ and $\gamma \equiv \gamma(0)$. The resemblance suggests the *Frobenius Ansatz* $y(t) = t^r \sum_{k=0}^{\infty} c_k t^k$ with $c_0 \neq 0$, a power of t times a power series.

Theorem 5.2 (Frobenius). *Suppose that β and γ are given by power series about 0 that converge for $|t| < \rho$, where $\rho \in (0, \infty]$, and that the roots r_1 and r_2 of the indicial equation*

$$r(r-1) + \beta(0)r + \gamma(0) = 0$$

are real, labeled so that $r_1 \geq r_2$. Then (18) has a fundamental set of solutions $\{y_1, y_2\}$ on $(0, \rho)$ of the following form, where every power series converges for $|t| < \rho$.

- (a) $y_1(t) = t^{r_1} \sum_{k=0}^{\infty} c_k t^k$ with $c_0 = 1$.
- (b) If $r_1 - r_2$ is not an integer, then $y_2(t) = t^{r_2} \sum_{k=0}^{\infty} d_k t^k$ with $d_0 = 1$.
- (c) If $r_1 = r_2$, then $y_2(t) = y_1(t) \ln t + t^{r_1} \sum_{k=1}^{\infty} d_k t^k$.
- (d) If $r_1 - r_2$ is a positive integer, then $y_2(t) = \kappa y_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} d_k t^k$ with $d_0 = 1$ and a constant $\kappa \in \mathbb{R}$, which may be zero.

On $(-\rho, 0)$, the same statements hold with t^{r_j} replaced by $|t|^{r_j}$ and $\ln t$ by $\ln |t|$.

Proof sketch. Write $\beta(t) = \sum_{k=0}^{\infty} \hat{\beta}_k t^k$ and $\gamma(t) = \sum_{k=0}^{\infty} \hat{\gamma}_k t^k$, and let $F(r) = r(r-1) + \hat{\beta}_0 r + \hat{\gamma}_0 = (r-r_1)(r-r_2)$. Substituting $y(t) = \sum_{k=0}^{\infty} c_k t^{k+r}$, we have $ty'(t) = \sum_{k=0}^{\infty} (k+r)c_k t^{k+r}$ and $t^2 y''(t) = \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k t^{k+r}$. Fact iv) and the identity theorem show that y solves (18) if and only if

$$F(r+k)c_k = - \sum_{j=0}^{k-1} \left((j+r)\hat{\beta}_{k-j} + \hat{\gamma}_{k-j} \right) c_j \quad \text{for every } k \geq 0.$$

For $k = 0$, this equation reads $F(r)c_0 = 0$, so $c_0 = 1$ forces $F(r) = 0$. For $r = r_1$, we have $F(r_1+k) = k(k+r_1-r_2) \neq 0$ for every $k \geq 1$, so the recurrence determines every c_k . For $r = r_2$, we have $F(r_2+k) = k(k-(r_1-r_2))$, which vanishes for some $k \geq 1$ if and only if $r_1 - r_2$ is a positive integer. So if $r_1 - r_2$ is not an integer, the recurrence determines every d_k . In both cases, $|F(r+k)|$ grows like k^2 , and a majorant argument as in the proof of Theorem 5.1 shows that the series converge for $|t| < \rho$. In case (b), the quotient y_2/y_1 behaves like $t^{r_2-r_1}$ as $t \rightarrow 0^+$, and in case (c) like $\ln t$. Neither quotient is constant near 0, so y_1 and y_2 are linearly independent. Example 5.3 below carries out the argument for case (c) by differentiating in r . We omit the proof of case (d); see, e.g., W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*. $\square$

5.2.3 Computing Frobenius series

Consider the ODE (Euler, 1769)

$$t^2(\alpha_0 + \alpha_1 t)y''(t) + t(\beta_0 + \beta_1 t)y'(t) + (\gamma_0 + \gamma_1 t)y(t) = 0, \quad (19)$$

where $\alpha_0 \neq 0$, α_1 , β_0 , β_1 , γ_0 , and γ_1 are given real numbers. The Euler equation (17) is the case $\alpha_1 = \beta_1 = \gamma_1 = 0$. Dividing (19) by $\alpha_0 + \alpha_1 t$ gives (18) with

$$\beta(t) = \frac{\beta_0 + \beta_1 t}{\alpha_0 + \alpha_1 t} \quad \text{and} \quad \gamma(t) = \frac{\gamma_0 + \gamma_1 t}{\alpha_0 + \alpha_1 t},$$

which are analytic for $|t| < \rho = |\alpha_0/\alpha_1|$ ($\rho = \infty$ if $\alpha_1 = 0$). Hence 0 is a regular singular point, and Theorem 5.2 applies. To compute, we insert $y(t) = t^r \sum_{k=0}^{\infty} c_k t^k$ directly into (19); comparing the coefficients of t^{k+r} gives the recurrence relations (R):

$$\chi(r)c_0 = 0 \quad (k = 0)$$

and

$$\chi(r+k)c_k = -((r+k-1)(r+k-2)\alpha_1 + (r+k-1)\beta_1 + \gamma_1)c_{k-1} \quad (k \geq 1),$$

where $\chi(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0$. We may assume that $c_0 \neq 0$, since if c_m is the first nonzero coefficient, we can replace r by $r+m$. The equation for $k=0$ then reduces to the indicial equation

$$\boxed{\chi(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 = \alpha_0 r^2 - (\alpha_0 - \beta_0)r + \gamma_0 = 0.}$$

Since $\beta(0) = \beta_0/\alpha_0$ and $\gamma(0) = \gamma_0/\alpha_0$, this equation is α_0 times the indicial equation of Theorem 5.2. To solve (19):

- i) Set $c_0 = 1$ and iterate (R) with $r = r_1$ to get the coefficients $c_k^{(1)}$ for $k \geq 1$ and hence $y_1(t) = t^{r_1} \sum_{k=0}^{\infty} c_k^{(1)} t^k$.
- ii) If $r_1 - r_2$ is not an integer, set $c_0 = 1$ again and iterate (R) with $r = r_2$ to get $c_k^{(2)}$ for $k \geq 1$ and hence $y_2(t) = t^{r_2} \sum_{k=0}^{\infty} c_k^{(2)} t^k$.
- iii) By Theorem 5.2 and Theorem 3.2, the general solution on $(0, \rho)$ is $y(t) = C_1 y_1(t) + C_2 y_2(t)$ with $C_1, C_2 \in \mathbb{R}$.

If $r_1 - r_2 = m$ is a positive integer, then $\chi(r_2 + m) = 0$, so (R) with $r = r_2$ and $k = m$ has a zero left-hand side. If its right-hand side also vanishes, then c_m is free and a second Frobenius series exists ($\kappa = 0$ in Theorem 5.2(d)); otherwise, no c_m satisfies (R), and a logarithmic term is required.

Example 5.2. Consider the ODE

$$2t^2 y''(t) - t y'(t) + (-t + 1)y(t) = 0.$$

Here $\alpha_0 = 2$, $\alpha_1 = 0$, $\beta_0 = -1$, $\beta_1 = 0$, $\gamma_0 = 1$, and $\gamma_1 = -1$, so $\rho = \infty$ and

$$\chi(r) = 2r^2 - 3r + 1 = 2(r-1)(r-\frac{1}{2}),$$

so $r_1 = 1$, $r_2 = 1/2$, and $r_1 - r_2 = 1/2$ is not an integer (Theorem 5.2(b)).

i) Recurrence with $r_1 = 1$: (R) reads $\chi(r_1 + k)c_k = c_{k-1}$ for $k \geq 1$, with $c_0 = 1$, so

$$c_k = \frac{c_{k-1}}{\chi(r_1 + k)} = \frac{c_{k-1}}{2(r_1 - 1 + k)(r_1 - \frac{1}{2} + k)} = \frac{c_{k-1}}{2k(k + \frac{1}{2})} = \frac{c_{k-2}}{2^2 k(k-1)(k + \frac{1}{2})(k - \frac{1}{2})} = \dots$$

So $c_1^{(1)} = \frac{1}{2 \cdot 1 \cdot (1 + \frac{1}{2})} = \frac{1}{3}$, $c_2^{(1)} = \frac{1/3}{2 \cdot 2 \cdot (2 + \frac{1}{2})} = \frac{1}{30}$, and in general

$$c_k^{(1)} = \frac{1}{2^k \left(\frac{2}{2} \cdot \frac{3}{2}\right) \left(\frac{4}{2} \cdot \frac{5}{2}\right) \left(\frac{6}{2} \cdot \frac{7}{2}\right) \dots \left(\frac{2k}{2} \cdot \frac{2k+1}{2}\right)} = \frac{1}{2^k 2^{-2k} (2k+1)!} = \frac{2^k}{(2k+1)!}.$$

Hence

$$y_1(t) = t \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!} t^k.$$

ii) Recurrence with $r_2 = \frac{1}{2}$: now $\chi(r_2 + k)c_k = c_{k-1}$, that is, $c_k = \frac{c_{k-1}}{2(k-\frac{1}{2})k}$, which gives $c_k^{(2)} = \frac{2^k}{(2k)!}$ and

$$y_2(t) = \sqrt{t} \sum_{k=0}^{\infty} \frac{2^k}{(2k)!} t^k.$$

iii) General solution on $(0, \infty)$:

$$y(t) = C_1 t \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!} t^k + C_2 \sqrt{t} \sum_{k=0}^{\infty} \frac{2^k}{(2k)!} t^k.$$

Example 5.3. Consider the ODE

$$t^2 y''(t) - t y'(t) + (1-t)y(t) = 0. \quad (20)$$

Here $\alpha_0 = 1$, $\beta_0 = -1$, $\gamma_0 = 1$, $\gamma_1 = -1$, and $\alpha_1 = \beta_1 = 0$, so $\chi(r) = r(r-1) - r + 1 = (r-1)^2$, which has the double zero $r_1 = r_2 = 1$ (Theorem 5.2(c), with $\rho = \infty$). With $c_0 = 1$, the recurrence is $\chi(r_1 + k)c_k = c_{k-1}$, where $\chi(r_1 + k) = k^2$. Hence $c_k = c_{k-1}/k^2 = 1/(k!)^2$, and

$$y_1(t) = t \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2}.$$

This gives *one* solution. Alas, we need two!

Second solution: we let r vary. Set $c_0 = 1$ and $\chi(r+k)c_k(r) = c_{k-1}(r)$ for $k \geq 1$, that is,

$$c_k(r) = \frac{1}{r^2} \cdot \frac{1}{(r+1)^2} \cdots \frac{1}{(r+k-1)^2}, \quad \text{and define} \quad z(t, r) := t^r \sum_{k=0}^{\infty} c_k(r) t^k.$$

Term-by-term differentiation gives

$$t \frac{\partial z}{\partial t}(t, r) = t^r \sum_{k=0}^{\infty} (k+r) c_k(r) t^k \quad \text{and} \quad t^2 \frac{\partial^2 z}{\partial t^2}(t, r) = t^r \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k(r) t^k.$$

Writing $\text{ODE}(z(t, r))$ for the left-hand side of (20) and shifting the index in the term $tz(t, r) = t^r \sum_{k=1}^{\infty} c_{k-1}(r) t^k$, we get

$$\text{ODE}(z(t, r)) = t^r \chi(r) + t^r \sum_{k=1}^{\infty} \underbrace{\left(((r+k)(r+k-1) - (r+k) + 1) c_k(r) - c_{k-1}(r) \right)}_{\chi(r+k)} t^k = t^r \chi(r),$$

because $\chi(r+k)c_k(r) - c_{k-1}(r) = 0$ by the recurrence. At $r = 1$, we recover y_1 .

Now differentiate both sides of $\text{ODE}(z(t, r)) = t^r \chi(r)$ with respect to r . Each $c_k(r)$ is analytic in r near $r_1 = 1$, and the series converges locally uniformly in (t, r) for $t > 0$, so we may differentiate term by term in r and interchange $\partial/\partial r$ with $\partial/\partial t$. On the right-hand side,

$$\frac{\partial}{\partial r} (t^r \chi(r)) = \chi(r) t^r \ln t + \chi'(r) t^r$$

vanishes at $r = 1$, since $\chi(1) = \chi'(1) = 0$. On the left-hand side, interchanging the derivatives gives $\text{ODE}\left(\frac{\partial z}{\partial r}(t, r)\right)\big|_{r=1} = 0$. Therefore,

$$y_2(t) := \frac{\partial z}{\partial r}(t, r)\big|_{r=1} = t \ln t \sum_{k=0}^{\infty} c_k(1) t^k + t \sum_{k=0}^{\infty} c'_k(1) t^k$$

is a second solution! It has the form of Theorem 5.2(c) with $d_k = c'_k(1)$, and $c'_0(1) = 0$ because $c_0 \equiv 1$. Since $c_k(r) > 0$ for r near 1, we may take logarithms:

$$\frac{c'_k(r)}{c_k(r)} = \frac{d}{dr} \ln(c_k(r)) = -2 \sum_{j=1}^k \frac{1}{r+j-1} \implies c'_k(1) = \frac{-2}{(k!)^2} \left(\sum_{j=1}^k \frac{1}{j} \right).$$

Therefore,

$$y_2(t) = -2t \sum_{k=0}^{\infty} \left(\frac{1}{(k!)^2} \left(\sum_{j=1}^k \frac{1}{j} \right) t^k \right) + t \ln t \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2}.$$

Because of the $t \ln t$ term, y_1 and y_2 are linearly independent, so the general solution on $(0, \infty)$ is $y(t) = C_1 y_1(t) + C_2 y_2(t)$. Phew!

Summary.

- 1) At an ordinary point t_0 of (16), every solution is a power series about t_0 . The series converges at least on the interval where the Taylor series of Q/P and R/P about t_0 both converge; for polynomial coefficients, it converges at least for $|t - t_0| < d$, where d is the distance from t_0 to the nearest complex zero of P . The coefficients $c_0 = y(t_0)$ and $c_1 = y'(t_0)$ are free, and the recurrence from comparing coefficients determines the others.

- 2) The Euler equation $\alpha_0 t^2 y'' + \beta_0 t y' + \gamma_0 y = 0$ has the solutions t^r , where r solves the indicial equation $\chi(r) = 0$; complex roots give $t^\sigma \cos(\tau \ln t)$ and $t^\sigma \sin(\tau \ln t)$, and a double root r_0 gives t^{r_0} and $t^{r_0} \ln t$.
- 3) At a regular singular point 0, the Frobenius Ansatz $y(t) = t^r \sum_{k=0}^{\infty} c_k t^k$ leads to the indicial equation $r(r-1) + \beta(0)r + \gamma(0) = 0$. The larger root r_1 always gives a Frobenius series solution.
- 4) The second solution is a Frobenius series when $r_1 - r_2$ is not an integer. It contains the term $y_1(t) \ln t$ when $r_1 = r_2$, and it may contain a multiple of this term when $r_1 - r_2$ is a positive integer.