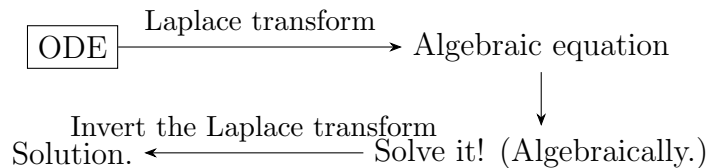


4 The Laplace transform

Section 3.3 promised a third method for inhomogeneous equations with constant coefficients: the Laplace transform. The idea is the following:



This method is often used in engineering and in the theory of Fourier transforms. In this chapter, we define the transform, derive its rules, and use them to solve linear ODEs with constant coefficients, including ODEs with discontinuous right-hand sides. We close the chapter with Green's functions.

4.1 Definition and examples

Definition 4.1. The *Laplace transform* takes a piecewise continuous function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ (or $\mathbb{C}$), where $\mathbb{R}_+ = [0, \infty)$, and maps it to a new function

$$\mathcal{L}[f](s) := F(s) = \int_0^\infty e^{-st} f(t) dt. \quad (\text{Notation: } f(t) \circ - \bullet F(s).)$$

Here *piecewise continuous* means that, on every bounded interval, f has at most finitely many discontinuities, each with finite one-sided limits. We denote transforms by capital letters: $F = \mathcal{L}[f]$, $Y = \mathcal{L}[y]$, and so on. For which s does the integral converge? The integrand $e^{-st} f(t)$ must decay, so f must not grow faster than an exponential.

Theorem 4.1. Suppose that f is piecewise continuous on $\mathbb{R}_+$ and that there exist constants $K > 0$, $\alpha \in \mathbb{R}$, and $M \geq 0$ such that

$$|f(t)| \leq K e^{\alpha t} \quad \forall t \geq M.$$

Then $\mathcal{L}[f](s)$ is well defined for every $s > \alpha$.

We then say that f is of *exponential order* α . For example, every bounded function is of exponential order 0, while e^{t^2} is not of exponential order.

Proof. We split the integral at $t = M$. The integral over $[0, M]$ is finite because a piecewise continuous function is bounded on $[0, M]$. For the remaining part,

$$\int_M^\infty e^{-st} |f(t)| dt \leq K \int_M^\infty e^{-(s-\alpha)t} dt < \infty$$

provided $s - \alpha > 0$. An absolutely convergent integral converges. □

Example 4.1. (1) $f(t) = 1$. Then $F(s) = \int_0^\infty e^{-st} dt = \frac{1}{s}$ for $s > 0$, so $\boxed{1 \circ - \bullet \frac{1}{s}}$.

(2) $f_n(t) = t^n$, and let $F_n(s) = \mathcal{L}[t^n](s)$. For $s > 0$ and $n \geq 1$, integration by parts gives

$$F_n(s) = \int_0^\infty e^{-st} t^n dt \stackrel{\text{I.B.P.}}{=} \underbrace{\left[-\frac{t^n e^{-st}}{s} \right]_0^\infty}_{=0} + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt = \frac{n}{s} F_{n-1}(s),$$

so $F_n(s) = \frac{n}{s} F_{n-1}(s) = \cdots = \frac{n!}{s^n} F_0(s) = \frac{n!}{s^{n+1}}$. Therefore, $\boxed{t^n \circ - \bullet \frac{n!}{s^{n+1}}}$ for $s > 0$.

(3) $f(t) = e^{\lambda t}$ with $\lambda \in \mathbb{C}$. Since $|e^{(\lambda-s)t}| = e^{(\operatorname{Re} \lambda - s)t} \rightarrow 0$ as $t \rightarrow \infty$ when $s > \operatorname{Re} \lambda$,

$$F(s) = \int_0^\infty e^{(\lambda-s)t} dt = \left[\frac{1}{\lambda-s} e^{(\lambda-s)t} \right]_0^\infty = \frac{1}{s-\lambda}.$$

Therefore, $\boxed{e^{\lambda t} \circ - \bullet \frac{1}{s-\lambda}}$ for $s > \operatorname{Re} \lambda$.

(4) $f(t) = \cos(\omega t)$ with $\omega \in \mathbb{R}$. Let $s > 0$. Integrating by parts twice, with $dv = e^{-st} dt$ both times,

$$F(s) = \underbrace{\left[-\frac{\cos(\omega t) e^{-st}}{s} \right]_0^\infty}_{=1/s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin(\omega t) dt = \frac{1}{s} - \frac{\omega}{s} \cdot \frac{\omega}{s} \underbrace{\int_0^\infty e^{-st} \cos(\omega t) dt}_{F(s)},$$

so $F(s) \left(1 + \frac{\omega^2}{s^2} \right) = \frac{1}{s}$, that is, $\boxed{\cos(\omega t) \circ - \bullet \frac{s}{s^2 + \omega^2}}$ for $s > 0$. Along the way, the sec-

ond integration by parts gives $\int_0^\infty e^{-st} \sin(\omega t) dt = \frac{\omega}{s} F(s)$, that is, $\boxed{\sin(\omega t) \circ - \bullet \frac{\omega}{s^2 + \omega^2}}$ for $s > 0$.

Remark 4.1. The Laplace transform is injective on piecewise continuous functions of exponential order (*Lerch's theorem*): if $\mathcal{L}[f] = \mathcal{L}[g]$, then $f = g$ at every point where both are continuous. This is what justifies recovering $y(t)$ from $Y(s)$. We write $f = \mathcal{L}^{-1}[F]$, or $F(s) \bullet - \circ f(t)$, and call f the *inverse Laplace transform* of F .

4.2 Rules

Theorem 4.2. Let f and g be piecewise continuous and of exponential order, with $f \circ - \bullet F$ and $g \circ - \bullet G$. Then the following rules hold for every sufficiently large s :

a) $c_1 f(t) + c_2 g(t) \circ - \bullet c_1 F(s) + c_2 G(s)$ for $c_1, c_2 \in \mathbb{R}$ (or $\mathbb{C}$).

b) $e^{\lambda t} f(t) \circ - \bullet F(s - \lambda)$ ($\lambda \in \mathbb{R}$).

c) $\begin{cases} 0, & t < c \\ f(t - c), & t \geq c \end{cases} \circ - \bullet e^{-cs} F(s)$ ($c > 0$).

d) $t^n f(t) \circ - \bullet (-1)^n F^{(n)}(s)$, where $F^{(n)}$ is the n th derivative.

e) $\int_0^t f(\tau) d\tau \circ - \bullet \frac{F(s)}{s}$.

f) $f'(t) \circ - \bullet sF(s) - f(0^+)$, provided that f is continuous on $(0, \infty)$ and f' is piecewise continuous. Here $f(0^+) := \lim_{t \rightarrow 0^+} f(t)$ is the right-hand limit of f at 0.

Proof. a) The integral is linear.

b) $\int_0^\infty e^{-st} e^{\lambda t} f(t) dt = \int_0^\infty e^{-(s-\lambda)t} f(t) dt = F(s - \lambda)$.

c) The substitution $\tau = t - c$ gives

$$\int_c^\infty e^{-st} f(t - c) dt = \int_0^\infty e^{-s(\tau+c)} f(\tau) d\tau = e^{-cs} F(s).$$

d) We differentiate under the integral sign (the growth condition justifies this step):

$$F'(s) = \int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t) dt = - \int_0^\infty e^{-st} t f(t) dt = -\mathcal{L}[tf](s).$$

Repeating the argument n times gives d).

f) With $u = e^{-st}$ and $dv = f'(t) dt$,

$$\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t) dt \stackrel{\text{I.B.P.}}{=} [e^{-st} f(t)]_{0^+}^\infty + s \int_0^\infty e^{-st} f(t) dt = -f(0^+) + sF(s),$$

where the growth condition ensures $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$. (If f' has jumps, we integrate by parts between consecutive jumps; the boundary terms cancel because f is continuous.)

e) The function $g(t) = \int_0^t f(\tau) d\tau$ is continuous, with $g' = f$ wherever f is continuous and $g(0) = 0$. By f), $F(s) = \mathcal{L}[g'](s) = sG(s)$. $\square$

Note: we can generalize f):

$$\mathcal{L}[f''](s) = s\mathcal{L}[f'](s) - f'(0^+) = s^2 F(s) - sf(0^+) - f'(0^+).$$

In general:

$$\boxed{\mathcal{L}[f^{(n)}](s) = s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - \dots - f^{(n-1)}(0^+).}$$

Combined with Example 4.1, the rules give the table below, where $n \in \mathbb{N}$ and $\omega \in \mathbb{R}$. For instance, rule d) with $n = 1$ gives

$$t \sin(\omega t) \circ - \bullet - \frac{d}{ds} \left(\frac{\omega}{s^2 + \omega^2} \right) = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

$f(t)$	$F(s)$	Valid for
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{\lambda t}, \lambda \in \mathbb{C}$	$\frac{n!}{(s - \lambda)^{n+1}}$	$s > \operatorname{Re} \lambda$
$\cos(\omega t), \sin(\omega t)$	$\frac{s}{s^2 + \omega^2}, \frac{\omega}{s^2 + \omega^2}$	$s > 0$
$e^{\lambda t} \cos(\omega t), e^{\lambda t} \sin(\omega t), \lambda \in \mathbb{R}$	$\frac{s - \lambda}{(s - \lambda)^2 + \omega^2}, \frac{\omega}{(s - \lambda)^2 + \omega^2}$	$s > \lambda$
$t \cos(\omega t), t \sin(\omega t)$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}, \frac{2\omega s}{(s^2 + \omega^2)^2}$	$s > 0$

4.3 Applications to ODEs

The idea is:

$$\begin{aligned} y(t) &\circ - \bullet Y(s), \\ y'(t) &\circ - \bullet sY(s) - y(0), \\ y''(t) &\circ - \bullet s^2Y(s) - sy(0) - y'(0). \end{aligned}$$

(The solution y is continuous at 0, so $y(0^+) = y(0)$.) For the initial value problem

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = y'_0,$$

with constant coefficients, linearity turns the ODE into the algebraic equation $(s^2 + a_1 s + a_0)Y(s) - (s + a_1)y_0 - y'_0 = F(s)$. The characteristic polynomial $p(s) = s^2 + a_1 s + a_0$ of Section 3.2 shows up again! Solving for Y ,

$$\boxed{Y(s) = \frac{(s + a_1)y_0 + y'_0 + F(s)}{p(s)}}. \quad (15)$$

To find y , we decompose Y into partial fractions and read off each inverse transform from the table. (The method assumes that y , y' , and y'' have Laplace transforms. This holds whenever f is continuous and of exponential order, as one can check with the variation of constants formula of Section 3.3.2.)

Example 4.2. Consider the ODE $y''(t) - y'(t) - 2y(t) = 0$ with $y(0) = 1$ and $y'(0) = 0$. Then (15) with $a_1 = -1$ gives

$$Y(s) = \frac{s - 1}{s^2 - s - 2} = \frac{s - 1}{(s - 2)(s + 1)} \stackrel{\text{partial fractions}}{=} \frac{A}{s - 2} + \frac{B}{s + 1},$$

where $A + B = 1$ and $A - 2B = -1$. Hence $B = 2/3$ and $A = 1/3$. But $\frac{1}{s-2} \bullet - \circ e^{2t}$ and $\frac{1}{s+1} \bullet - \circ e^{-t}$. Therefore,

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Example 4.3. Consider the ODE $y'(t) + 3y(t) = \cos t$ with $y(0) = 0$. Taking the Laplace transform,

$$sY(s) + 3Y(s) = \frac{s}{s^2 + 1} \implies Y(s) = \frac{s}{(s + 3)(s^2 + 1)}.$$

Partial fractions give

$$Y(s) = \frac{1}{10} \left[\frac{3s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{3}{s + 3} \right],$$

and therefore

$$y(t) = \frac{1}{10} [3 \cos t + \sin t - 3e^{-3t}].$$

Example 4.4 (Resonance, revisited). Let's solve the resonant oscillator of Section 3.5 once more:

$$y''(t) + \omega_0^2 y(t) = \frac{F_0}{m} \cos(\omega_0 t), \quad y(0) = y'(0) = 0.$$

By (15),

$$Y(s) = \frac{F_0}{m} \frac{s}{(s^2 + \omega_0^2)^2}.$$

The table gives $t \sin(\omega_0 t) \circ - \bullet \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$ (this is where we need rule d)), so

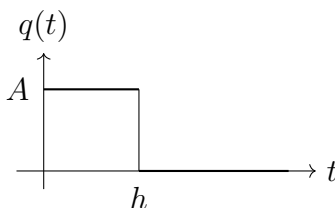
$$y(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t),$$

as before, but without guessing a trial form.

4.4 Discontinuous functions

Example 4.5. Consider an impulse of magnitude A :

$$q(t) = \begin{cases} A, & 0 \leq t < h, \\ 0, & t \geq h, \end{cases} \quad h > 0.$$



What is its Laplace transform? We have

$$\mathcal{L}[q](s) = \int_0^h A e^{-st} dt = \left[-\frac{A}{s} e^{-st} \right]_0^h = \frac{A}{s} [1 - e^{-sh}].$$

Definition 4.2. The function

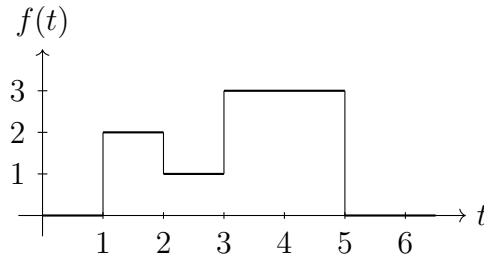
$$t \mapsto H(t) := \begin{cases} 1, & t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is called the *Heaviside function*. Its Laplace transform is $\mathcal{L}[H](s) = 1/s$.

In general, we express step functions (piecewise constants) in terms of Heaviside functions. For instance, $q(t) = A(H(t) - H(t - h))$. With Heaviside functions, rule c) of Theorem 4.2 reads

$$H(t - c)f(t - c) \circ - \bullet e^{-cs}F(s), \quad \text{so that} \quad H(t - c) \circ - \bullet \frac{e^{-cs}}{s}.$$

Example 4.6. Consider the staircase function sketched below.



Each jump contributes a shifted Heaviside function times the height of the jump: $f(t) = 2H(t - 1) - H(t - 2) + 2H(t - 3) - 3H(t - 5)$. Hence

$$F(s) = \frac{2}{s}e^{-s} - \frac{1}{s}e^{-2s} + \frac{2}{s}e^{-3s} - \frac{3}{s}e^{-5s}.$$

Example 4.7. Consider the ODE

$$y'(t) + 3y(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & t \geq 1, \end{cases} \quad y(0) = 0,$$

that is, $y'(t) + 3y(t) = H(t) - H(t - 1)$. Taking the Laplace transform,

$$sY(s) + 3Y(s) = \frac{1}{s} - \frac{1}{s}e^{-s} \implies Y(s) = R(s) - e^{-s}R(s), \quad R(s) = \frac{1}{s(s+3)}.$$

Partial fractions give

$$R(s) = \frac{1}{3} \left[\frac{1}{s} - \frac{1}{s+3} \right] \bullet - \circ \frac{1}{3} [1 - e^{-3t}] = r(t),$$

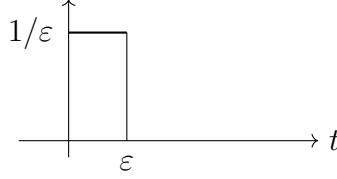
and rule c) gives $e^{-s}R(s) \bullet - \circ H(t - 1)r(t - 1)$. Therefore,

$$y(t) = r(t) - H(t - 1)r(t - 1) = \begin{cases} \frac{1}{3}(1 - e^{-3t}), & 0 \leq t < 1, \\ \frac{1}{3}(e^{-3(t-1)} - e^{-3t}), & t \geq 1. \end{cases}$$

Note: y is continuous, but y' jumps at $t = 1$, where the right-hand side jumps. The ODE holds at every $t \neq 1$.

Special case: the Dirac δ function. For $\varepsilon > 0$, define the impulse of magnitude $1/\varepsilon$ and duration ε ,

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}H(t) - \frac{H(t - \varepsilon)}{\varepsilon}.$$



We then “define” $\delta_0 = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon$ (a distribution, or generalized function). The Dirac δ -function δ_0 gives “unit mass” at $t = 0$: for a continuous function f ,

$$\int_0^\infty f(t)\delta_0(t) dt \text{ “=” } \lim_{\varepsilon \rightarrow 0} \underbrace{\frac{1}{\varepsilon} \int_0^\varepsilon f(t) dt}_{\text{average over } (0, \varepsilon)} = f(0).$$

Formally, the Laplace transform of the Dirac δ -function is

$$\mathcal{L}[\delta_0](s) \text{ “=” } \lim_{\varepsilon \rightarrow 0} \mathcal{L}[\delta_\varepsilon](s) = \lim_{\varepsilon \rightarrow 0} \frac{1 - e^{-\varepsilon s}}{\varepsilon s} = 1,$$

because $\frac{1-e^{-x}}{x} \rightarrow \frac{d}{dx}e^x \Big|_{x=0} = 1$ as $x \rightarrow 0$. Therefore,

$$\boxed{\delta_0(t) \circ - \bullet 1.}$$

We will see why this is useful in Section 4.6 on Green’s functions.

4.5 Convolution rule

Definition 4.3. For two functions f and g , the *convolution* $f * g$ is given by:

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau = \int_0^t f(\tau)g(t - \tau) d\tau = (g * f)(t).$$

Theorem 4.3. Let f and g be piecewise continuous, with $|f(t)| \leq Ke^{\alpha t}$ and $|g(t)| \leq Ke^{\alpha t}$ for every $t \geq 0$, where $K > 0$ and $\alpha \in \mathbb{R}$. Then $f * g$ is well defined and

$$\boxed{(f * g)(t) \circ - \bullet F(s)G(s)} \quad (s > \alpha),$$

where $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$.

Proof. Step 1. We have

$$|(f * g)(t)| \leq \int_0^t |f(t - \tau)||g(\tau)| d\tau \leq \int_0^t Ke^{\alpha(t-\tau)}Ke^{\alpha\tau} d\tau = K^2te^{\alpha t}.$$

Hence $f * g$ is well defined, and $\mathcal{L}[f * g](s)$ exists for $s > \alpha$.

Step 2. For $s > \alpha$,

$$\mathcal{L}[f * g](s) = \int_0^\infty \int_0^t (e^{-s\tau} g(\tau)) (e^{-s(t-\tau)} f(t-\tau)) d\tau dt.$$

By Step 1, this double integral converges absolutely, so we may interchange the order of integration (Fubini–Tonelli theorem): integrating τ from 0 to t and t from 0 to ∞ is the same as integrating t from τ to ∞ and τ from 0 to ∞ . With $u = t - \tau$ in the inner integral,

$$\begin{aligned} \mathcal{L}[f * g](s) &= \int_0^\infty e^{-s\tau} g(\tau) \left(\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau) dt \right) d\tau \\ &= \underbrace{\left(\int_0^\infty e^{-s\tau} g(\tau) d\tau \right)}_{G(s)} \underbrace{\left(\int_0^\infty e^{-su} f(u) du \right)}_{F(s)}. \end{aligned}$$

□

Example 4.8. Let $f(t) = \sqrt{t}$. What is its Laplace transform? We have

$$(f * f)(t) = \int_0^t \sqrt{(t-\tau)\tau} d\tau.$$

Let $\tau = \frac{t}{2} + x$, $d\tau = dx$ (so that $-\frac{t}{2} \leq x \leq \frac{t}{2}$):

$$(f * f)(t) = \int_{-t/2}^{t/2} \sqrt{\frac{t^2}{4} - x^2} dx \stackrel{\text{trig. substitution}}{=} \frac{\pi t^2}{8}.$$

But $(f * f)(t) \circ - \bullet F(s)^2$, so

$$F(s)^2 = \frac{\pi}{8} \mathcal{L}[t^2](s) = \frac{\pi}{8} \cdot \frac{2!}{s^3}.$$

Since $F(s) > 0$ (the integrand $e^{-st}\sqrt{t}$ is positive), we find $\sqrt{t} \circ - \bullet \frac{\sqrt{\pi}}{2s^{3/2}}$ for $s > 0$.

4.6 Green's functions

Example 4.9. Consider $y'(t) + ay(t) = f(t)$ with $y(0) = 0$. Taking the Laplace transform,

$$sY(s) + aY(s) = F(s) \implies Y(s) = \underbrace{\left(\frac{1}{s+a} \right)}_{:=G(s)} F(s),$$

where the factor $G(s)$ depends on the homogeneous part of the ODE and the factor $F(s)$ depends on $f(t)$. We call $G(s)$ the *transfer function* of the (homogeneous part of the) ODE.

In general, consider

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = f(t), \quad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

where $a_0, a_1, \dots, a_{n-1}$ are constant coefficients. All initial values vanish, so

$$Y(s) = \frac{1}{p(s)}F(s),$$

where p is the characteristic polynomial (Remark 3.1), $F = \mathcal{L}[f]$, and $G = \frac{1}{p}$ is the transfer function.

Definition 4.4. We call the inverse Laplace transform $g = \mathcal{L}^{-1}[G]$ of the transfer function the *Green's function* of the ODE. It depends only on the left-hand side of the ODE, not on f .

Partial fractions write $1/p(s)$ as a sum of terms $c/(s - \lambda)^k$, where λ is a root of p , so the table shows that g exists and is a combination of the functions $t^{k-1}e^{\lambda t}$ of Remark 3.1.

Special case: suppose f is the Dirac δ -function δ_0 . Then $\mathcal{L}[\delta_0](s) = 1$, and in that case, $Y(s) = \frac{1}{p(s)} = G(s)$. So the Green's function is the response of the system, initially at rest, to a unit impulse at $t = 0$.

Back to the general case: $Y(s) = G(s)F(s)$ (convolution!). By Theorem 4.3 and Lerch's theorem, we find

$$Y(s) = G(s)F(s) \quad \bullet - \circ \quad \boxed{y(t) = (g * f)(t) = \int_0^t g(t - \tau)f(\tau) d\tau.}$$

We can solve the ODE (find $y(t)$) using the following procedure:

- (1) Compute the Green's function $g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t)$, where p is the characteristic polynomial of the ODE.
- (2) Solve the ODE: $y(t) = (g * f)(t)$ for any inhomogeneous term f .

Hence we compute the Green's function once for a given ODE and then obtain any solution from step (2).

Example 4.10 (Harmonic oscillator). Consider

$$y''(t) + \omega_0^2 y(t) = f(t), \quad y(0) = y'(0) = 0.$$

- (1) Find the Green's function: the characteristic polynomial is $p(s) = s^2 + \omega_0^2$, and

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega_0^2}\right] = \frac{\sin(\omega_0 t)}{\omega_0} = g(t).$$

- (2) Then, for $t \geq 0$,

$$y(t) = (g * f)(t) = \frac{1}{\omega_0} \int_0^t \sin(\omega_0(t - \tau))f(\tau) d\tau.$$

Summary.

- 1) The Laplace transform $F(s) = \int_0^\infty e^{-st} f(t) dt$ exists for $s > \alpha$ when f is piecewise continuous and of exponential order α , and it determines f at its points of continuity (Lerch's theorem).
- 2) The transform turns derivatives into multiplication by s , up to initial values. An initial value problem with constant coefficients becomes an algebraic equation for $Y(s)$, with the characteristic polynomial as denominator; partial fractions and the table invert it.
- 3) Step functions are combinations of shifted Heaviside functions, and the Dirac δ -function formally has transform 1.
- 4) The transform of a convolution is the product of the transforms.
- 5) The Green's function $g = \mathcal{L}^{-1}[1/p]$ depends only on the left-hand side of the ODE, and the solution with zero initial values is $y = g * f$.