MATH 441: Differential Equations
Exercises for Chapter 6

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1 Exercises (Linear systems of ODEs)

Exercise 1.1. Consider the coupled mass–spring system of Example 6.1 of the notes with $m_1 = m_2 = 1$, $k_1 = k_2 = k_3 = 1$, and no external forces.

a) Write the equations of motion as a first-order system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ for $\boldsymbol{y} = (x_1, x_2, v_1, v_2)^\top$, where $v_j = x_j'$.

b) Show that the eigenvalues of $\boldsymbol{A}$ are $\pm i$ and $\pm i\sqrt{3}$, and find the corresponding eigenvectors. **Hint:** The Ansatz $\boldsymbol{y} = e^{\lambda t}\boldsymbol{v}$ with $\boldsymbol{v} = (\boldsymbol{u}, \lambda\boldsymbol{u})^\top$ and $\boldsymbol{u} \in \mathbb{R}^2$ reduces the problem to the 2×2 eigenvalue problem $\boldsymbol{K}\boldsymbol{u} = -\lambda^2\boldsymbol{u}$ for the stiffness matrix $\boldsymbol{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

c) Write the general real solution. Describe the two motions (*normal modes*) in which the masses oscillate with a single frequency: how do x_1 and x_2 move relative to each other in each mode?

d) Find the solution with $x_1(0) = 1$, $x_2(0) = 0$, and $v_1(0) = v_2(0) = 0$. Is this motion periodic?

Solutions:

Solutions (continued):

Exercise 1.2. Let $\boldsymbol{\Phi}$ be a fundamental matrix of $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$ on an open interval I, where $\boldsymbol{A}$ is continuous.

a) Show that $\boldsymbol{\Phi M}$ is a fundamental matrix for every invertible constant matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, and that every fundamental matrix has this form. **Hint:** For the second claim, use Theorem 6.2 of the notes.
b) Show that $t \mapsto \boldsymbol{\Phi}(t)^{-1}$ is differentiable on I with $\frac{d}{dt}\boldsymbol{\Phi}(t)^{-1} = -\boldsymbol{\Phi}(t)^{-1}\boldsymbol{A}(t)$. **Hint:** Differentiate $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(t)^{-1} = \boldsymbol{I}$. Deduce that $\frac{\partial}{\partial s}\boldsymbol{R}(t,s) = -\boldsymbol{R}(t,s)\boldsymbol{A}(s)$ for the resolvent.
c) Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Compute $e^{\boldsymbol{A}}$, $e^{\boldsymbol{B}}$, and $e^{\boldsymbol{A}}e^{\boldsymbol{B}}$, and show that $e^{\boldsymbol{A}}e^{\boldsymbol{B}} \neq e^{\boldsymbol{B}}e^{\boldsymbol{A}}$. Deduce that $e^{\boldsymbol{A}+\boldsymbol{B}}$ equals at most one of these two products, so that Proposition 6.5(a) of the notes fails without the commutation hypothesis. (For the curious: $\boldsymbol{A}+\boldsymbol{B}$ has eigenvalues ± 1, and $e^{\boldsymbol{A}+\boldsymbol{B}} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$.)

Solutions:

Solutions (continued):

Exercise 1.3. Consider the two systems

$$\text{(i)}\ \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ -2y_2(t) \end{pmatrix}, \qquad \text{(ii)}\ \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ -y_1(t) \end{pmatrix}.$$

a) For each system, find a fundamental matrix $\boldsymbol{\Phi}(t)$ and compute the resolvent $\boldsymbol{R}(t, t_0)$. Verify that $\boldsymbol{R}(t, s)\boldsymbol{R}(s, t_0) = \boldsymbol{R}(t, t_0)$ in each case.
b) Compute the Wronskian $W(t) = \det \boldsymbol{\Phi}(t)$ for each system and check the Abel–Liouville identity.
c) Use Duhamel's formula (Theorem 6.3 of the notes) to solve system (ii) with the added forcing $\boldsymbol{f}(t) = (0, 1)^\top$ and the initial value $\boldsymbol{y}(0) = \boldsymbol{0}$. Check your answer by substitution.
d) Solve system (ii) with the forcing $\boldsymbol{f}(t) = (0, \cos t)^\top$ and $\boldsymbol{y}(0) = \boldsymbol{0}$, and explain the growth of the solution in terms of Section 3.5 of the notes.

Solutions:

Solutions (continued):

Exercise 1.4. Consider the matrix

$$\boldsymbol{A} = \frac{1}{4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

a) Show that $\boldsymbol{A}$ is not diagonalizable.
b) Compute $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, and $\boldsymbol{A}^4$, guess a formula for $\boldsymbol{A}^k$, prove it by induction, and use the definition of the matrix exponential to compute $e^{\boldsymbol{A}t}$.
c) Compute $e^{\boldsymbol{A}t}$ again by writing $\boldsymbol{A} = \lambda_0 \boldsymbol{I} + \boldsymbol{N}$ with $\boldsymbol{N}^2 = \boldsymbol{0}$, as in Section 6.3.2 of the notes, and compare with b).
d) Find the general solution of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, classify the equilibrium at the origin, and sketch the phase portrait, indicating the eigenvector direction.

Solutions:

Solutions (continued):

Exercise 1.5. Let $\alpha \in \mathbb{R}$, and consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} \alpha + 2 & 1 \\ -4 & \alpha - 2 \end{pmatrix}.$$

a) Show that $\boldsymbol{A}$ cannot be diagonalized for any choice of $\alpha \in \mathbb{R}$.
b) For $t \in \mathbb{R}$, compute the matrix exponential $e^{\boldsymbol{A}t}$. Your result should be expressed in terms of elementary functions.
c) Use b) to find the solution of the initial value problem $\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t)$, $\boldsymbol{y}(0) = (1, 0)^\top$, and identify the resolvent matrix of this system.
d) For which values of α is the origin asymptotically stable? Lyapunov stable? Unstable? Justify your answer with Theorem 6.7 of the notes, and explain what happens at $\alpha = 0$.

Solutions:

Solutions (continued):

Exercise 1.6. Consider the system $x'(t) = 5x - 2y$, $y'(t) = 17x - 5y$.

a) Write the system as $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix}$, and find the eigenvalues and eigenvectors of $\boldsymbol{A}$.
b) Find the general real solution of the system.
c) Classify the equilibrium at the origin.
d) Show that $17x^2 - 10xy + 2y^2$ is constant along every solution, and sketch the phase portrait.

Solutions:

Solutions (continued):

Exercise 1.7. Sketch the phase portrait and classify the equilibrium at the origin for each of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

a) $x'(t) = x + 3y, \quad y'(t) = 3x + y.$
b) $x'(t) = 4x - 5y, \quad y'(t) = 5x - 4y.$
c) $x'(t) = 3x + 2y, \quad y'(t) = x + 4y.$
d) (Love affairs.) In the model of Example 6.13 of the notes, Romeo is a "narcissistic nerd" and Juliet is a "bashful budder" who responds to Romeo's feelings:

$$R'(t) = R(t) - 2J(t), \qquad J'(t) = R(t) - J(t).$$

Classify the equilibrium and describe the fate of the romance.

Solutions:

Solutions (continued):