

MATH 441: Differential Equations

Exercises for Chapter 5

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1 Exercises (Series solutions of second-order ODEs)

Exercise 1.1. Consider the ODE

$$\frac{d^2y}{dt^2} + y = 0.$$

- a)** Explain why $t_0 = 0$ is an ordinary point, and why every series solution about $t_0 = 0$ converges for every $t \in \mathbb{R}$ (Theorem 5.1 of the notes).
- b)** Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^k$ into the ODE and show that $c_{k+2} = -c_k / ((k+2)(k+1))$ for every $k \geq 0$.
- c)** Find closed-form expressions for c_{2m} and c_{2m+1} in terms of c_0 and c_1 . Deduce that the general solution is $y(t) = c_0 \cos t + c_1 \sin t$, and compare with the harmonic oscillator in Section 3.2 of the notes.

Solutions:

Solutions (continued):

Exercise 1.2. Consider the ODE

$$(t^2 + 1)\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0.$$

a) Show that $t_0 = 0$ is an ordinary point, and that Theorem 5.1 of the notes guarantees that every series solution about $t_0 = 0$ converges at least for $|t| < 1$.

b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^k$ into the ODE and show that

$$c_{k+2} = -\frac{(k-2)(k-3)}{(k+2)(k+1)}c_k \quad \text{for every } k \geq 0.$$

c) Deduce that every solution is a polynomial of degree at most 3, and find the general solution.

d) The series in c) converge for every $t \in \mathbb{R}$, while a) guarantees convergence only for $|t| < 1$. Explain why this does not contradict Theorem 5.1 of the notes.

Solutions:

Solutions (continued):

Exercise 1.3. Let $\nu \in \mathbb{R}$, and consider the Legendre equation

$$(1 - t^2) \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + \nu(\nu + 1)y = 0.$$

- a) Show that $t_0 = 0$ is an ordinary point, and that Theorem 5.1 of the notes guarantees that every series solution about $t_0 = 0$ converges at least for $|t| < 1$.
b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^k$ into the ODE and show that

$$c_{k+2} = \frac{(k - \nu)(k + \nu + 1)}{(k + 2)(k + 1)} c_k \quad \text{for every } k \geq 0.$$

- c) Let y_1 and y_2 be the series solutions with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, respectively. Write y_1 up to the term in t^4 and y_2 up to the term in t^5 .
d) Suppose that one of these two series does not terminate. Apply the ratio test to its nonzero terms to show that its radius of convergence is exactly 1.
e) Let $\nu = n$ be a nonnegative integer. Show that one of y_1 and y_2 is a polynomial of degree n . Compute this polynomial for $n = 2$ and for $n = 3$.

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.4. Let $\alpha_0 \neq 0$, β_0 , and γ_0 be real constants, and consider the Euler equation

$$\alpha_0 t^2 \frac{d^2 y}{dt^2} + \beta_0 t \frac{dy}{dt} + \gamma_0 y = 0, \quad t > 0.$$

a) Find the general solution on $(0, \infty)$ of each ODE. Express the solution in (a3) without complex numbers.

$$(a1) \quad 2t^2 \frac{d^2 y}{dt^2} + 3t \frac{dy}{dt} - 15y = 0. \quad (a2) \quad t^2 \frac{d^2 y}{dt^2} - 3t \frac{dy}{dt} + 4y = 0. \quad (a3) \quad t^2 \frac{d^2 y}{dt^2} + 3t \frac{dy}{dt} + 4y = 0.$$

b) Let y be twice differentiable on $(0, \infty)$, and define $u(x) = y(e^x)$ for $x \in \mathbb{R}$. Show that y solves the Euler equation if and only if u solves

$$\alpha_0 \frac{d^2 u}{dx^2} + (\beta_0 - \alpha_0) \frac{du}{dx} + \gamma_0 u = 0,$$

and that the characteristic polynomial of this equation is the indicial polynomial χ of Section 5.2 of the notes. How do the three cases for the Euler equation correspond to the three cases of Section 3.2?

c) Use b) and the method of undetermined coefficients to find the general solution on $(0, \infty)$ of each ODE.

$$(c1) \quad t^2 \frac{d^2 y}{dt^2} - t \frac{dy}{dt} + y = 1. \quad (c2) \quad t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2y = 3t^2 + \ln t.$$

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.5. Consider the ODE

$$4t \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y = 0.$$

- a)** Show that $t_0 = 0$ is a regular singular point. Write the ODE in the form $t^2 y'' + t\beta(t)y' + \gamma(t)y = 0$ of Theorem 5.2 of the notes, and show that the roots of the indicial equation are $r_1 = 1/2$ and $r_2 = 0$. Which case of Theorem 5.2 applies?
- b)** Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^{k+r}$ into the ODE and show that $2r(2r-1)c_0 = 0$ and

$$2(k+r)(2k+2r-1)c_k = -c_{k-1} \quad \text{for every } k \geq 1.$$

- c)** Let $c_0 = 1$. Find closed-form expressions for the coefficients c_k when $r = r_1$ and when $r = r_2$. Show that the two resulting Frobenius series are $\sin \sqrt{t}$ and $\cos \sqrt{t}$.
- d)** For which $\alpha \in \mathbb{R}$ does the ODE have a solution on $(0, \infty)$ with $\lim_{t \rightarrow 0^+} y(t) = 1$ and $\lim_{t \rightarrow 0^+} y'(t) = \alpha$?

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.6. Let n be a nonnegative integer, and consider the Bessel equation

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2)y = 0.$$

a) Show that $t_0 = 0$ is a regular singular point, and find the roots of the indicial equation. Which case of Theorem 5.2 (Frobenius) applies when $n = 0$? When $n \geq 1$?

b) Substitute $y(t) = \sum_{k=0}^{\infty} c_k t^{k+r}$ into the ODE and show that $(r^2 - n^2)c_0 = 0$, $((r+1)^2 - n^2)c_1 = 0$, and

$$((r+k)^2 - n^2)c_k = -c_{k-2} \quad \text{for every } k \geq 2.$$

c) Let $r = n$. Show that every odd coefficient vanishes and that

$$c_{2m} = \frac{(-1)^m n!}{4^m m! (m+n)!} c_0 \quad \text{for every } m \geq 0.$$

With the normalization $c_0 = 1/(2^n n!)$, deduce that the corresponding solution is the Bessel function of the first kind of order n ,

$$J_n(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{t}{2}\right)^{2m+n}.$$

d) Let $n \geq 1$ and $r = -n$, with $c_0 = 1$. Show that the recurrence in b) cannot hold at $k = 2n$. What does this imply about the constant κ in Theorem 5.2(d)?

Solutions:

Solutions (continued):

Solutions (continued):