

MATH 441: Differential Equations

Exercises for Chapter 4

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1 Exercises (The Laplace transform)

Exercise 1.1. Let $\omega, c \in \mathbb{R}$. Compute the Laplace transform of each function, and state the values of s for which it is defined.

a) $\sinh(\omega t)$ and $\cosh(\omega t)$. (Hint: write both functions with exponentials and use Example 4.1 of the notes.)

b) te^{ct} and t^2e^{ct} . Use rule d) of Theorem 4.2 of the notes, and check your answers with rule b).

c) The function f defined by $f(t) = t$ for $0 \leq t < 1$ and $f(t) = 0$ for $t \geq 1$. (Hint: write f with Heaviside functions, and write $t = (t - 1) + 1$.)

Solutions:

Solutions (continued):

Exercise 1.2. a) Show that, for every $s \in \mathbb{R}$, the integral $\int_0^T e^{-st} e^{t^2} dt$ tends to ∞ as $T \rightarrow \infty$. Conclude that e^{t^2} has no Laplace transform.

b) Let f be piecewise continuous and of exponential order α , with $f(t) \circ - \bullet F(s)$, and let $c > 0$. Show that

$$f(ct) \circ - \bullet \frac{1}{c} F\left(\frac{s}{c}\right) \quad \text{for every } s > c\alpha.$$

c) Let $\omega > 0$. Use b) and $\cos t \circ - \bullet \frac{s}{s^2+1}$ to recover the transform of $\cos(\omega t)$.

Solutions:

Solutions (continued):

Exercise 1.3. Find the inverse Laplace transform of each function.

a) $\frac{1}{s^2 + 2s}$. b) $\frac{4s^3}{s^4 - 4}$. c) $\frac{1}{s(s^2 + 1)}$. d) $\frac{s}{(s + 1)^2}$.

e) $\frac{s + 3}{s^2 + 2s + 5}$. f) $\frac{e^{-2s}}{s^2 + 1}$.

g) $\frac{1}{(s^2 + 1)^2}$. (Hint: compute the convolution $(\sin * \sin)(t)$ and use Theorem 4.3 of the notes.)

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.4. Use the Laplace transform to solve the following initial value problems.

a) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$, with $y(0) = 1$ and $y'(0) = 0$.

b) $\frac{d^2y}{dt^2} + y = g(t)$, with $y(0) = 1$ and $y'(0) = 0$, where

$$g(t) = \begin{cases} 0 & \text{if } 0 \leq t < 3\pi, \\ 1 & \text{if } t \geq 3\pi. \end{cases}$$

What is the amplitude of the oscillation for $t < 3\pi$? For $t \geq 3\pi$?

Solutions:

Solutions (continued):

Exercise 1.5. Consider the initial value problem

$$\frac{d^2y}{dt^2} + 4y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \quad \text{where } f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi, \\ 0 & \text{if } t \geq \pi. \end{cases}$$

a) Write f with Heaviside functions, and show that

$$Y(s) = \frac{s}{s^2 + 4} + \frac{1 - e^{-\pi s}}{s(s^2 + 4)}.$$

b) Find $y(t)$ for every $t \geq 0$, and show that $y(t) = \cos(2t)$ for every $t \geq \pi$.

c) Check that y and y' are continuous at $t = \pi$, and find the jump of y'' at $t = \pi$, that is, $\lim_{t \rightarrow \pi^+} y''(t) - \lim_{t \rightarrow \pi^-} y''(t)$.

Solutions:

Solutions (continued):

Exercise 1.6. Let $\alpha > 0$, and consider the initial value problem

$$\frac{d^2y}{dt^2} - \alpha^2 y = \sin(\alpha t), \quad y(0) = 0, \quad y'(0) = 0.$$

- a) Find the transfer function and the Green's function of the ODE. (Hint: use Exercise ?? a).)
- b) Write the solution of the initial value problem as a convolution integral.
- c) Compute $Y(s)$, and invert it to evaluate the integral in b) in closed form. (Hint: $\frac{1}{(s^2 - \alpha^2)(s^2 + \alpha^2)} = \frac{1}{2\alpha^2} \left(\frac{1}{s^2 - \alpha^2} - \frac{1}{s^2 + \alpha^2} \right)$.)
- d) Let f be continuous and of exponential order. Show that the variation-of-constants formula of Section 3.3 of the notes, with $t_0 = 0$ and the fundamental set $\{e^{\alpha t}, e^{-\alpha t}\}$, gives the same particular solution of $y'' - \alpha^2 y = f(t)$ as the convolution of the Green's function with f .

Solutions:

Solutions (continued):

Solutions (continued):