

MATH 441: Assignment 5

Instructions: Submit your *handwritten* solutions in class by *Friday, October 30th, 11 AM*. I will *not* accept typed solutions unless you have an accommodation.

Readings

Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima.

Boyce and DiPrima: Sections 6.1–6.6

Problem 1

Let $\omega, c \in \mathbb{R}$. Compute the Laplace transform of each function, and state the values of s for which it is defined.

- a) $\sinh(\omega t)$ and $\cosh(\omega t)$. **Hint:** Write both functions with exponentials and use Example 4.1 of the notes.
- b) te^{ct} and t^2e^{ct} . Use rule d) of Theorem 4.2 of the notes, and check your answers with rule b).
- c) $e^{-2t}t \cos(3t)$.
- d) The function f defined by $f(t) = t$ for $0 \leq t < 1$ and $f(t) = 0$ for $t \geq 1$. **Hint:** Write f with Heaviside functions, and write $t = (t - 1) + 1$.
- e) $t^2H(t - 1)$, where H is the Heaviside function.

Problem 2

Find the inverse Laplace transform of each function.

- a) $\frac{2s + 1}{s^2 + 4s + 13}$.
- b) $\frac{1}{s^2(s + 1)}$.
- c) $\frac{e^{-\pi s}}{s(s^2 + 1)}$.

- d) $F(s) = \ln \left(\frac{s+1}{s-1} \right)$, for $s > 1$. You may assume that F is the Laplace transform of a piecewise continuous function f of exponential order. **Hint:** Use rule d) of Theorem 4.2 of the notes to find $tf(t)$.

Problem 3

Use the Laplace transform to solve each initial value problem.

- a) $y''(t) - 4y'(t) + 4y(t) = te^{2t}$, with $y(0) = 1$ and $y'(0) = 0$.
b) $y'''(t) - y''(t) + y'(t) - y(t) = 0$, with $y(0) = 1$ and $y'(0) = y''(0) = 0$.
c) $y''(t) + 2y'(t) + 2y(t) = g(t)$, with $y(0) = 0$ and $y'(0) = 1$, where

$$g(t) = \begin{cases} 1, & \pi \leq t < 2\pi, \\ 0, & 0 \leq t < \pi \text{ or } t \geq 2\pi. \end{cases}$$

Problem 4

Let $k \in \mathbb{R}$ and $T > 0$. A mass on a spring starts at equilibrium with unit velocity and receives an impulse of magnitude k at time T :

$$y''(t) + y(t) = k\delta_0(t - T), \quad y(0) = 0, \quad y'(0) = 1.$$

- a) Rule c) of Theorem 4.2 of the notes and $\delta_0(t) \circ - \bullet 1$ give, formally, $\delta_0(t - T) \circ - \bullet e^{-Ts}$. Use this rule to show that $y(t) = \sin t + kH(t - T)\sin(t - T)$. Show that y is continuous at $t = T$ and that y' jumps by k at $t = T$.
b) Find all pairs (T, k) such that $y(t) = 0$ for every $t \geq T$.
c) Let $\varepsilon > 0$, and let y_ε solve the same initial value problem with $k\delta_0(t - T)$ replaced by $k\delta_\varepsilon(t - T) = \frac{k}{\varepsilon}(H(t - T) - H(t - T - \varepsilon))$. Find $y_\varepsilon(t)$ for $t \geq T + \varepsilon$, and show that $\lim_{\varepsilon \rightarrow 0} y_\varepsilon(t) = y(t)$ for every $t > T$.

Problem 5

- a) Compute the convolution $(\sin * \cos)(t)$ directly from the definition. Then check your answer with Theorem 4.3 of the notes.
b) Use the Laplace transform to solve each integral equation. You may assume that the solution y is continuous and of exponential order.

$$(1) \quad y(t) + \int_0^t (t - \tau)y(\tau) d\tau = 1;$$

$$(2) \quad y(t) = t + \int_0^t \sin(t - \tau)y(\tau) d\tau.$$

- c) Let $m, n \in \mathbb{N}$. Use Theorem 4.3 of the notes to show that

$$\int_0^t \tau^m (t - \tau)^n d\tau = \frac{m! n!}{(m + n + 1)!} t^{m+n+1} \quad \forall t \geq 0.$$

$$\text{Deduce that } \int_0^1 x^m (1 - x)^n dx = \frac{m! n!}{(m + n + 1)!}.$$

Problem 6

Consider the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

- a) Find the transfer function and the Green's function of the ODE.
- b) Let f be continuous and of exponential order. Write the solution of the initial value problem as a convolution integral.
- c) Let $f(t) = e^{-t}$. Evaluate the integral in b), and check your answer by computing $Y(s)$ and inverting it.
- d) Let $f(t) = e^{-t}$ again, but now impose $y(0) = 1$ and $y'(0) = -1$. Use equation (15) of the notes to show that the solution is the sum of the solution of the homogeneous equation with these initial values and the solution found in c). Find the solution.

Problem 7

Let f be piecewise continuous and of exponential order α , with $f(t) \circ - \bullet F(s)$.

- a) Show that $\lim_{s \rightarrow \infty} F(s) = 0$. **Hint:** Split the integral at $t = M$, as in the proof of Theorem 4.1 of the notes. Conclude that no piecewise continuous function of exponential order has Laplace transform 1.
- b) Suppose, in addition, that f is continuous on $(0, \infty)$ and that f' is piecewise continuous and of exponential order. Prove the *initial value theorem*:

$$\lim_{s \rightarrow \infty} sF(s) = f(0^+).$$

Check this formula for $f(t) = \cos(\omega t)$.

- c) Suppose that $F(s) = \frac{2s^2 + 1}{(s + 1)(s^2 + 1)}$ is the Laplace transform of a function f that satisfies the hypotheses of b). Find $f(0^+)$ without computing f . Then compute f and check your answer.

Problem 8

The Bessel function of the first kind of order 0, denoted J_0 , solves the initial value problem

$$ty''(t) + y'(t) + ty(t) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

You may assume that J_0 , J'_0 , and J''_0 are continuous and bounded on $[0, \infty)$. Let $Y = \mathcal{L}[J_0]$.

- a) Use rules d) and f) of Theorem 4.2 of the notes to show that $(s^2 + 1)Y'(s) + sY(s) = 0$ for every sufficiently large s .
- b) Solve the ODE in a) to show that $Y(s) = \frac{c}{\sqrt{s^2 + 1}}$ for some constant c .
- c) Use Problem 7 b) to show that $c = 1$, that is, $J_0(t) \circ - \bullet \frac{1}{\sqrt{s^2 + 1}}$.