

MATH 441: Assignment 4

Instructions: Submit your *handwritten* solutions in class by *Friday, October 16th, 11 AM*. I will *not* accept typed solutions unless you have an accommodation.

Readings

Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima.

Boyce and DiPrima: Sections 3.1, 3.3–3.8, and 4.2

Problem 1

Find the general solution of each ODE. In a), also solve the initial value problem.

- a) $y''(t) + 2y'(t) + 5y(t) = 0$, with $y(0) = 1$ and $y'(0) = -3$.
- b) $y^{(5)}(t) + 3y^{(4)}(t) - y'''(t) - 7y''(t) + 4y(t) = 0$. **Hint:** Verify that the characteristic polynomial equals $(\lambda - 1)^2(\lambda + 2)^2(\lambda + 1)$.
- c) The equation *Septimi Gradus* of Leonhard Euler (1743):

$$y^{(7)}(t) + y^{(5)}(t) + y^{(4)}(t) + y'''(t) + y''(t) + y(t) = 0.$$

Hint: Verify that the characteristic polynomial equals $(\lambda + 1)(\lambda^2 + \lambda + 1)(\lambda^2 - \lambda + 1)^2$.

Problem 2

In this problem, you will prove the claim left as an exercise in Remark 3.1 of the notes, for real roots. Consider the n th-order equation $L[y] = 0$, where

$$L[y](t) = y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t)$$

with real constants $a_0, \dots, a_{n-1}$, and let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be its characteristic polynomial. Let $\lambda_0 \in \mathbb{R}$ be a root of p of multiplicity m .

- a) Show that $L[e^{\lambda t}] = p(\lambda)e^{\lambda t}$ for every $\lambda \in \mathbb{R}$.

- b) Differentiate the identity in a) k times with respect to λ to show that

$$L[t^k e^{\lambda t}] = \sum_{j=0}^k \binom{k}{j} p^{(j)}(\lambda) t^{k-j} e^{\lambda t}.$$

(You may interchange derivatives with respect to t and λ .)

- c) Write $p(\lambda) = (\lambda - \lambda_0)^m q(\lambda)$ to show that $p^{(j)}(\lambda_0) = 0$ for $j = 0, 1, \dots, m-1$. Conclude that $e^{\lambda_0 t}, t e^{\lambda_0 t}, \dots, t^{m-1} e^{\lambda_0 t}$ solve $L[y] = 0$.
- d) Show that these m functions are linearly independent on $\mathbb{R}$.

Problem 3

Use the method of undetermined coefficients in each part.

- a) Find the general solution of $y''(t) - 4y(t) = t e^{-t}$.
- b) Find the general solution of $y''(t) + 4y(t) = t^2 + 3e^t + 3 \sin(2t)$. **Hint:** Find a particular solution for each of the three terms on the right-hand side separately.
- c) Solve the initial value problem $y''(t) - 2y'(t) + y(t) = e^t$, with $y(0) = 1$ and $y'(0) = 0$.

Problem 4

Use variation of constants in a) and b).

- a) Find the general solution of

$$t y''(t) - (t+1) y'(t) + y(t) = t^2, \quad t > 0.$$

Hint: In Problem 7 of Assignment 3, you showed that $\{1+t, e^t\}$ is a fundamental set of solutions of the homogeneous equation on $(0, \infty)$.

- b) Find the general solution of

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2}, \quad t \in \mathbb{R}.$$

- c) Explain why the method of undetermined coefficients does not apply to the equation in b).

Problem 5

- a) Let $a_1, a_0 \in \mathbb{R}$. Prove that every solution of $y''(t) + a_1 y'(t) + a_0 y(t) = 0$ satisfies $\lim_{t \rightarrow \infty} y(t) = 0$ if and only if $a_1 > 0$ and $a_0 > 0$.

Now let $m > 0$, $k > 0$, and $\gamma \geq 0$, and let y solve the damped oscillator equation $my''(t) + \gamma y'(t) + ky(t) = 0$.

- b) Show that the energy $E(t) = \frac{1}{2}my'(t)^2 + \frac{1}{2}ky(t)^2$ satisfies $E'(t) = -\gamma y'(t)^2$ for every $t \in \mathbb{R}$. Conclude that E is nonincreasing, and that E is constant when $\gamma = 0$.
- c) Suppose that the oscillator is critically damped, $\gamma^2 = 4km$, and let $\omega_0 = \sqrt{k/m}$. Show that a nonzero solution vanishes at most once. Then suppose that $y(0) = y_0 > 0$ and $y'(0) = v_0$, and show that the mass passes through the equilibrium position at some time $t > 0$ if and only if $v_0 < -\omega_0 y_0$.

Problem 6

Consider the forced oscillator

$$y''(t) + 9y(t) = 5 \cos(\omega t), \quad y(0) = 0, \quad y'(0) = 0.$$

- a) Let $\omega = 2$. Solve the initial value problem, and write the solution as a product of two sines.
- b) Let $\omega = 3$. Solve the initial value problem, and describe the behavior of the solution as $t \rightarrow \infty$.
- c) Now add damping: consider $y''(t) + y'(t) + 9y(t) = 5 \cos(\omega t)$. Find the amplitude $R(\omega)$ of the steady-state solution, the frequency $\omega_{\max}$ at which R is largest, and the value $R(\omega_{\max})$.

Problem 7

Let $\omega_0 > 0$ and $T > 0$, and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and T -periodic, that is, $f(t + T) = f(t)$ for every $t \in \mathbb{R}$. Consider the equation

$$y''(t) + \omega_0^2 y(t) = f(t), \quad t \in \mathbb{R}.$$

- a) Let y be a solution. Show that y is T -periodic if and only if $y(T) = y(0)$ and $y'(T) = y'(0)$. **Hint:** Show that $z(t) = y(t + T)$ is also a solution, and use Theorem 3.1 of the notes.
- b) Show that the variation-of-constants formula of Section 3.3.2 of the notes, with the fundamental set $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$ and $t_0 = 0$, gives the particular solution

$$y_p(t) = \frac{1}{\omega_0} \int_0^t \sin(\omega_0(t - \tau)) f(\tau) d\tau,$$

and that $y_p(0) = y'_p(0) = 0$. **Hint:** $\frac{d}{dt} \int_0^t K(t, \tau) d\tau = K(t, t) + \int_0^t \frac{\partial K}{\partial t}(t, \tau) d\tau$ when K and $\frac{\partial K}{\partial t}$ are continuous.

- c) Write every solution as $y = y_p + c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. Use a) to show that y is T -periodic if and only if (c_1, c_2) solves a linear system of two equations whose determinant equals $2\omega_0 (1 - \cos(\omega_0 T))$. Conclude that if $\omega_0 T$ is not an integer multiple of 2π , then the equation has exactly one T -periodic solution.
- d) Let $f(t) = \cos t$ and $T = 2\pi$. Find all 2π -periodic solutions in each of the cases $\omega_0 = \sqrt{2}$, $\omega_0 = 2$, and $\omega_0 = 1$.

Problem 8

Prove Theorem 3.2 of the notes in your own words. Let I be an open interval, let a_1 and a_0 be continuous on I , and consider the homogeneous equation

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0, \quad t \in I.$$

Theorem (Structure of the solution set). (a) *The solutions of the homogeneous equation form a vector space of dimension two.*

(b) *If y_1 and y_2 are linearly independent solutions of the homogeneous equation, then every solution y can be written as $y = c_1 y_1 + c_2 y_2$ for unique constants $c_1, c_2 \in \mathbb{R}$.*

You may use Theorem 3.1 (existence and uniqueness), Proposition 3.1 (superposition principle), Lemma 3.1, and Corollary 3.1 of the notes without proof. Justify every step.

Problem 9

A function $f: [0, \infty) \rightarrow \mathbb{R}$ is *piecewise continuous* if, on every bounded interval, f has at most finitely many discontinuities, each with finite one-sided limits. Let f be piecewise continuous, and suppose that there exist constants $K > 0$, $\alpha \in \mathbb{R}$, and $M \geq 0$ such that

$$|f(t)| \leq K e^{\alpha t} \quad \forall t \geq M.$$

For $s \in \mathbb{R}$, consider the improper integral

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

- a) Prove that $F(s)$ is well defined, that is, the improper integral converges, for every $s > \alpha$. You may use without proof that an improper integral $\int_0^\infty g(t) dt$ of a piecewise continuous function g converges whenever $\int_0^\infty |g(t)| dt$ converges.
- b) For each of the following functions, find the values of α for which the bound above holds (for suitable K and M), and compute $F(s)$ for every s at which a) guarantees convergence:
- (1) $f(t) = 1$;
 - (2) $f(t) = t^n$, with $n \in \mathbb{N}$ (**Hint:** integrate by parts and use induction on n);
 - (3) $f(t) = e^{\lambda t}$, with $\lambda \in \mathbb{C}$ (here $|\cdot|$ denotes the modulus of a complex number, and the result in a) holds for complex-valued f as well);
 - (4) $f(t) = \cos(\omega t)$ and $f(t) = \sin(\omega t)$, with $\omega \in \mathbb{R}$.