

MATH 441: Solutions to Assignment 3

Problem 1

Consider the initial value problem

$$(3t^2y(t) + 2ty(t) + y(t)^3) + (t^2 + y(t)^2) y'(t) = 0, \quad y(0) = 1.$$

Write $M(t, y) = 3t^2y + 2ty + y^3$ and $N(t, y) = t^2 + y^2$.

- a) Show that the equation is not exact.
- b) Let $\mu = \mu(t)$ be a nonvanishing differentiable function of t alone. Show that, at points where $N(t, y) \neq 0$, the equation $\mu M + \mu N y' = 0$ is exact if and only if

$$\mu'(t) = \mu(t) \frac{\frac{\partial M}{\partial y}(t, y) - \frac{\partial N}{\partial t}(t, y)}{N(t, y)}.$$

Verify that the quotient depends on t only, and find an integrating factor $\mu(t)$.

- c) Find a potential function V for the exact equation obtained in b), and solve the initial value problem in implicit form.

Solutions: **a)** We have $\frac{\partial M}{\partial y} = 3t^2 + 2t + 3y^2$ and $\frac{\partial N}{\partial t} = 2t$. These differ, for example at $(t, y) = (0, 1)$, where they equal 3 and 0. Hence the equation is not exact.

b) Since μ depends on t only, the product rule gives

$$\frac{\partial(\mu M)}{\partial y} = \mu \frac{\partial M}{\partial y}, \quad \frac{\partial(\mu N)}{\partial t} = \mu' N + \mu \frac{\partial N}{\partial t}.$$

The new equation is exact if and only if these are equal, that is, $\mu' N = \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \right)$. Where $N \neq 0$, dividing by N gives the stated condition. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} = \frac{3t^2 + 3y^2}{t^2 + y^2} = 3,$$

which depends on t only (it is constant). The condition reads $\mu' = 3\mu$, so $\mu(t) = e^{3t}$ is an integrating factor.

c) We look for V with $\frac{\partial V}{\partial y} = e^{3t}(t^2 + y^2)$ and $\frac{\partial V}{\partial t} = e^{3t}(3t^2y + 2ty + y^3)$. Integrating the first equation in y gives $V(t, y) = e^{3t} \left(t^2y + \frac{y^3}{3} \right) + g(t)$. Then

$$\frac{\partial V}{\partial t} = 3e^{3t} \left(t^2y + \frac{y^3}{3} \right) + 2te^{3t}y + g'(t) = e^{3t} (3t^2y + 2ty + y^3) + g'(t),$$

so $g' = 0$, and we take $g = 0$:

$$V(t, y) = e^{3t} \left(t^2y + \frac{y^3}{3} \right).$$

Along a solution, $V(t, y(t))$ is constant, and $V(0, 1) = \frac{1}{3}$. Multiplying by 3, the solution satisfies

$$e^{3t} (3t^2y(t) + y(t)^3) = 1.$$

Problem 2

Let $r > 0$ and $K > 0$.

- a) Solve the logistic equation

$$N'(t) = rN(t) \left(1 - \frac{N(t)}{K} \right), \quad N(0) = N_0 > 0.$$

Hint: You can either solve it directly using partial fractions or make the change of variable $x(t) = 1/N(t)$ and then derive and solve the corresponding ODE for $x(t)$.

- b) Compute $\lim_{t \rightarrow +\infty} N(t)$ from your solution, and check that the answer agrees with the phase line of the logistic equation in the notes.

Suppose now that the population is harvested at a constant rate $H > 0$:

$$N'(t) = rN(t) \left(1 - \frac{N(t)}{K} \right) - H.$$

- c) Suppose that $0 < H < rK/4$. Find the two equilibria $N_- < N_+$, draw the phase line, and classify each equilibrium as stable or unstable. What happens to the population if $N_0 > N_-$? If $0 < N_0 < N_-$?
- d) Suppose that $H > rK/4$. Show that $N'(t) \leq rK/4 - H < 0$ as long as $N(t) \geq 0$, and conclude that the population reaches 0 in finite time, whatever the value of $N_0 > 0$.

Solutions: a) Let $x = 1/N$. Then

$$x'(t) = -\frac{N'(t)}{N(t)^2} = -\frac{r}{N(t)} + \frac{r}{K} = -rx(t) + \frac{r}{K}, \quad x(0) = \frac{1}{N_0}.$$

This equation is linear. Multiplying by e^{rt} gives $(e^{rt}x(t))' = \frac{r}{K}e^{rt}$, and integrating from 0 to t gives

$$x(t) = \frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K} \right) e^{-rt}.$$

Therefore

$$N(t) = \frac{1}{x(t)} = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}, \quad t \geq 0.$$

The denominator equals $N_0(1 - e^{-rt}) + Ke^{-rt} > 0$ for $t \geq 0$, so $N(t)$ is defined and positive for every $t \geq 0$.

b) Since $e^{-rt} \rightarrow 0$ as $t \rightarrow +\infty$, we have $N(t) \rightarrow KN_0/N_0 = K$. This agrees with the phase line in the notes: 0 is unstable, K is stable, and every solution with $N_0 > 0$ tends to K .

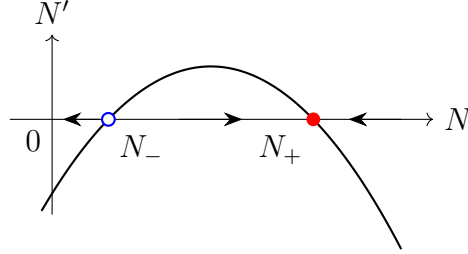
c) Write $f(N) = rN \left(1 - \frac{N}{K} \right) - H$. Completing the square gives

$$f(N) = \frac{rK}{4} - H - \frac{r}{K} \left(N - \frac{K}{2} \right)^2. \quad (1)$$

The equilibria are the roots of f ,

$$N_{\pm} = \frac{K}{2} \left(1 \pm \sqrt{1 - \frac{4H}{rK}} \right),$$

which are real and distinct because $0 < H < rK/4$, and satisfy $0 < N_- < N_+ < K$. The graph of f is a downward parabola, so $f < 0$ for $N < N_-$, $f > 0$ for $N_- < N < N_+$, and $f < 0$ for $N > N_+$. The phase line is



The arrows point away from N_- and toward N_+ , so N_- is unstable and N_+ is stable.

If $N_0 > N_-$, then $N(t) \rightarrow N_+$: the population increases to N_+ if $N_- < N_0 < N_+$, decreases to N_+ if $N_0 > N_+$, and stays at N_+ if $N_0 = N_+$.

If $0 < N_0 < N_-$, then N decreases. By (1), f is increasing for $N < K/2$, so $N'(t) = f(N(t)) \leq f(N_0) < 0$ as long as $N(t) \leq N_0$. The population therefore decreases at least at the rate $|f(N_0)|$ and reaches 0 by the time $N_0/|f(N_0)|$: it dies out in finite time.

d) By (1), $f(N) \leq rK/4 - H$ for every N , so $N'(t) \leq rK/4 - H < 0$ as long as $N(t) \geq 0$. Let $\delta = H - rK/4 > 0$. Integrating from 0 to t gives

$$N(t) \leq N_0 - \delta t$$

as long as $N(t) \geq 0$. The right-hand side vanishes at $t = N_0/\delta$, so the population reaches 0 no later than $t = N_0/(H - rK/4)$, whatever the value of $N_0 > 0$.

Problem 3

The proof of the Picard–Lindelöf theorem (Theorem 2.2 in the notes) is constructive. You’ll explore its main idea in this problem. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

with f continuous. Starting from the constant function $\varphi_0(t) = y_0$, define the *Picard iterates*

$$\varphi_{k+1}(t) = y_0 + \int_{t_0}^t f(\tau, \varphi_k(\tau)) d\tau, \quad k = 0, 1, 2, \dots$$

Under assumptions (A1)–(A3) in Theorem 2.2 of the notes, the iterates converge on $[t_0, T]$ to the unique solution of the initial value problem. We will prove this later in the course.

- a) Show that a continuously differentiable function y solves the initial value problem on $[t_0, T]$ if and only if it satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) d\tau \quad \text{for every } t \in [t_0, T].$$

(Thus a solution is a fixed point of the map $\varphi \mapsto y_0 + \int_{t_0}^t f(\tau, \varphi(\tau)) d\tau$.)

Now consider the initial value problem

$$y'(t) = 1 - 2ty(t), \quad y(0) = 0.$$

- b) Let $T = 1/2$. Show that assumptions (A1), (A2), and (A3) hold on the rectangle D_T for every $C > 0$, with $K = 1$ and $L = 1$. Choose C so that Theorem 2.2 guarantees a unique solution on $[0, 1/2]$.
- c) Compute the Picard iterates φ_1 , φ_2 , and φ_3 .
- d) Use variation of constants to show that the solution is

$$y(t) = e^{-t^2} \int_0^t e^{s^2} ds.$$

(Do not attempt to evaluate this integral.)

- e) Expand e^{s^2} and e^{-t^2} in Taylor series to show that the Taylor polynomial of degree 5 of the solution in d) equals $\varphi_3(t)$.

Solutions: a) Suppose that y solves the initial value problem. By the fundamental theorem of calculus, for every $t \in [t_0, T]$,

$$y(t) = y(t_0) + \int_{t_0}^t y'(\tau) d\tau = y_0 + \int_{t_0}^t f(\tau, y(\tau)) d\tau.$$

Conversely, suppose that y satisfies the integral equation. Setting $t = t_0$ gives $y(t_0) = y_0$. The integrand $\tau \mapsto f(\tau, y(\tau))$ is continuous, so the fundamental theorem of calculus gives $y'(t) = f(t, y(t))$ for every $t \in [t_0, T]$.

- b) Here $f(t, y) = 1 - 2ty$, $t_0 = 0$, $y_0 = 0$, and $D_T = [0, 1/2] \times [-C, C]$.

(A1) The function f is a polynomial, hence continuous on D_T .

(A2) For every $t \in [0, 1/2]$, $|f(t, 0)| = 1$, so (A2) holds with $K = 1$.

(A3) For every $t \in [0, 1/2]$ and $y_1, y_2 \in [-C, C]$, $|f(t, y_1) - f(t, y_2)| = 2t|y_1 - y_2| \leq |y_1 - y_2|$, so (A3) holds with $L = 1$.

These bounds hold for every $C > 0$. Theorem 2.2 requires $C \geq \frac{K}{L} (e^{L(T-t_0)} - 1) = e^{1/2} - 1 \approx 0.65$, so we may take $C = 1$. Theorem 2.2 then gives a unique solution on $[0, 1/2]$.

c) With $\varphi_0(t) = 0$, we find

$$\begin{aligned}\varphi_1(t) &= \int_0^t 1 \, d\tau = t, \\ \varphi_2(t) &= \int_0^t (1 - 2\tau^2) \, d\tau = t - \frac{2t^3}{3}, \\ \varphi_3(t) &= \int_0^t \left(1 - 2\tau^2 + \frac{4\tau^4}{3}\right) \, d\tau = t - \frac{2t^3}{3} + \frac{4t^5}{15}.\end{aligned}$$

d) The homogeneous equation $y' = -2ty$ has the solutions $y(t) = Le^{-t^2}$. Let $y(t) = L(t)e^{-t^2}$. Then

$$y'(t) + 2ty(t) = L'(t)e^{-t^2} - 2tL(t)e^{-t^2} + 2tL(t)e^{-t^2} = L'(t)e^{-t^2},$$

so $y' + 2ty = 1$ if and only if $L'(t) = e^{t^2}$. The initial condition gives $L(0) = y(0) = 0$, so $L(t) = \int_0^t e^{s^2} \, ds$ and

$$y(t) = e^{-t^2} \int_0^t e^{s^2} \, ds.$$

e) From $e^u = 1 + u + \frac{u^2}{2} + \dots$, we get

$$e^{s^2} = 1 + s^2 + \frac{s^4}{2} + \dots, \quad e^{-t^2} = 1 - t^2 + \frac{t^4}{2} - \dots.$$

Integrating the first series term by term gives

$$\int_0^t e^{s^2} \, ds = t + \frac{t^3}{3} + \frac{t^5}{10} + \dots.$$

Multiplying the two series and keeping the terms of degree at most 5,

$$y(t) = \left(1 - t^2 + \frac{t^4}{2}\right) \left(t + \frac{t^3}{3} + \frac{t^5}{10}\right) + \dots = t + \left(\frac{1}{3} - 1\right)t^3 + \left(\frac{1}{10} - \frac{1}{3} + \frac{1}{2}\right)t^5 + \dots.$$

Since $\frac{1}{3} - 1 = -\frac{2}{3}$ and $\frac{1}{10} - \frac{1}{3} + \frac{1}{2} = \frac{4}{15}$, the Taylor polynomial of degree 5 of y is $t - \frac{2t^3}{3} + \frac{4t^5}{15} = \varphi_3(t)$.

Problem 4

Let $t > 0$ and $N \in \mathbb{N}$, and apply Euler's method with step size $h = t/N$ to the initial value problem

$$y'(s) = y(s), \quad y(0) = 1,$$

whose solution is $y(s) = e^s$.

- a) Show that after N steps, Euler's method gives $y_N = (1 + t/N)^N$.
- b) Using the inequalities $x - \frac{x^2}{2} \leq \ln(1 + x) \leq x$ for every $x \geq 0$, deduce that

$$0 \leq e^t - \left(1 + \frac{t}{N}\right)^N \leq \frac{t^2 e^t}{2N}.$$

Hint: Write $(1 + t/N)^N = e^{N \ln(1+t/N)}$, and use $e^{-u} \geq 1 - u$ for $u \geq 0$.

- c) Rewrite the bound in b) in terms of h , and explain why it shows that Euler's method has order one.

Solutions: **a)** Here $f(s, y) = y$, so Euler's method reads $y_{n+1} = y_n + h y_n = (1 + h)y_n$, with $y_0 = 1$. By induction, $y_n = (1 + h)^n$. With $h = t/N$ and $n = N$, we obtain $y_N = (1 + t/N)^N$.

- b)** Apply the inequalities with $x = t/N > 0$ and multiply by N :

$$t - \frac{t^2}{2N} \leq N \ln \left(1 + \frac{t}{N}\right) \leq t.$$

The exponential function is increasing, so $(1 + t/N)^N = e^{N \ln(1+t/N)} \leq e^t$, which is the left inequality. For the right inequality, let $u = t^2/(2N) \geq 0$. Then

$$\left(1 + \frac{t}{N}\right)^N \geq e^{t-u} = e^t e^{-u} \geq e^t (1 - u), \quad \text{so} \quad e^t - \left(1 + \frac{t}{N}\right)^N \leq e^t u = \frac{t^2 e^t}{2N}.$$

- c)** Since $N = t/h$, the bound reads

$$0 \leq e^t - y_N \leq \frac{te^t}{2}h.$$

For a fixed final time t , the factor $te^t/2$ does not depend on h , so the error is at most a constant times h : halving the step size halves the bound. This is what it means for Euler's method to have order one.

Problem 5

Let $\lambda > 0$, and consider the initial value problem

$$y'(t) = -\lambda y(t), \quad y(0) = y_0 \neq 0.$$

- a) Solve the problem, and compute $\lim_{t \rightarrow \infty} y(t)$.
- b) Apply Euler's method with step size $h > 0$, and find a formula for y_n . For which values of h does the sequence $\{y_n\}_{n=0}^{\infty}$ converge to 0? For which values of h is it bounded? For which values of h do all the y_n have the same sign as y_0 ?
- c) The *implicit* (or backward) Euler method evaluates f at the new point instead of the old one:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

Apply it to this problem, solve for y_{n+1} , and show that $y_n \rightarrow 0$ as $n \rightarrow \infty$ for every step size $h > 0$.

Note: Parts b) and c) illustrate the *stability* of numerical methods, which we will study in the chapter on numerical methods.

Solutions: a) The solution is $y(t) = y_0 e^{-\lambda t}$, and $y(t) \rightarrow 0$ as $t \rightarrow \infty$ because $\lambda > 0$.

b) Euler's method reads $y_{n+1} = y_n - h\lambda y_n = (1 - h\lambda)y_n$, so

$$y_n = (1 - h\lambda)^n y_0, \quad n = 0, 1, 2, \dots$$

This is a geometric sequence with ratio $q = 1 - h\lambda$.

- It converges to 0 if and only if $|q| < 1$, that is, $0 < h < 2/\lambda$.
- It is bounded if and only if $|q| \leq 1$, that is, $0 < h \leq 2/\lambda$. For $h = 2/\lambda$, we get $y_n = (-1)^n y_0$.
- All y_n have the sign of y_0 if and only if $q > 0$, that is, $0 < h < 1/\lambda$. For $h = 1/\lambda$, we get $y_n = 0$ for $n \geq 1$, and for $h > 1/\lambda$, the sign alternates.

c) The implicit Euler method reads $y_{n+1} = y_n - h\lambda y_{n+1}$, so

$$y_{n+1} = \frac{y_n}{1 + h\lambda} \quad \text{and} \quad y_n = \frac{y_0}{(1 + h\lambda)^n}.$$

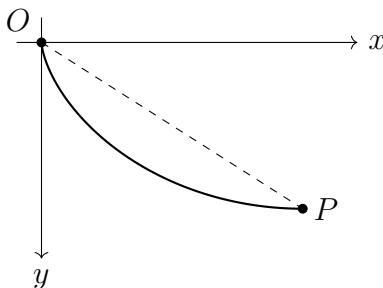
For every $h > 0$, we have $1 + h\lambda > 1$, so $y_n \rightarrow 0$ as $n \rightarrow \infty$.

Problem 6 (The brachistochrone)

In 1696, Johann Bernoulli asked for the curve along which a bead, sliding without friction under gravity, travels from a point O to a lower point P in the shortest time. This curve is called the *brachistochrone*, from the Greek for “shortest time.” Place O at the origin, measure x to the right and y *downward*, and describe the curve as the graph of a function $y = y(x)$ with $y(0) = 0$. Bernoulli showed that the brachistochrone satisfies the ODE

$$y(x) (1 + y'(x)^2) = k^2$$

for some constant $k > 0$. You may take this fact as given.



- a) Show that $0 < y(x) \leq k^2$ at every point x where the ODE holds, and that on an interval where $y' > 0$, the ODE is equivalent to the separable equation

$$y'(x) = \sqrt{\frac{k^2 - y(x)}{y(x)}}.$$

- b) Separate the variables and use the substitution $y = k^2 \sin^2 \phi$, with $0 \leq \phi \leq \pi/2$, to show that the solution with $y(0) = 0$ is given in parametric form by

$$x = k^2 (\phi - \sin \phi \cos \phi), \quad y = k^2 \sin^2 \phi.$$

Hint: $\sin^2 \phi = \frac{1}{2} (1 - \cos(2\phi))$.

- c) Set $\theta = 2\phi$ and $a = k^2/2$. Show that the curve in b) is the *cycloid*

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta),$$

the curve traced by a point on a circle of radius a that rolls along the x -axis. Check directly that $\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta}$ along the cycloid, and that the cycloid satisfies the ODE $y(1 + y'^2) = k^2$.

- d) Let $P = (\pi a, 2a)$, which corresponds to $\theta = \pi$. The travel time from O to P along a curve $y = y(x)$ is proportional to the (improper, but convergent) integral

$$J[y] = \int_0^{\pi a} \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}} dx.$$

(You may take this fact as given.) Show that $J = \pi\sqrt{2a}$ along the cycloid and $J = \sqrt{\pi^2 + 4}\sqrt{2a}$ along the straight line from O to P (dashed in the figure). Which curve is faster? **Hint:** Along the cycloid, write the integral in terms of θ , using $\sqrt{1 + y'(x)^2} dx = \sqrt{x'(\theta)^2 + y'(\theta)^2} d\theta$ and $1 - \cos \theta = 2 \sin^2(\theta/2)$.

Solutions: a) Since $1 + y'(x)^2 \geq 1$ and $k^2 > 0$, the ODE gives

$$y(x) = \frac{k^2}{1 + y'(x)^2} \in (0, k^2].$$

Solving the ODE for y'^2 gives $y'(x)^2 = \frac{k^2 - y(x)}{y(x)}$. Where $y' > 0$, we take the positive square root and obtain the separable equation. Conversely, squaring the separable equation and multiplying by y gives back the ODE.

b) Separating the variables gives

$$\int \sqrt{\frac{y}{k^2 - y}} dy = \int 1 dx.$$

With $y = k^2 \sin^2 \phi$, we have $dy = 2k^2 \sin \phi \cos \phi d\phi$ and, for $0 < \phi < \pi/2$,

$$\sqrt{\frac{y}{k^2 - y}} = \sqrt{\frac{\sin^2 \phi}{\cos^2 \phi}} = \frac{\sin \phi}{\cos \phi}.$$

The left-hand integral becomes

$$\int 2k^2 \sin^2 \phi d\phi = \int k^2 (1 - \cos(2\phi)) d\phi = k^2 \left(\phi - \frac{\sin(2\phi)}{2} \right) + c = k^2 (\phi - \sin \phi \cos \phi) + c.$$

Hence $x = k^2 (\phi - \sin \phi \cos \phi) + c$. The condition $y(0) = 0$ means $\phi = 0$ when $x = 0$, so $c = 0$, and

$$x = k^2 (\phi - \sin \phi \cos \phi), \quad y = k^2 \sin^2 \phi.$$

c) With $\theta = 2\phi$ and $k^2 = 2a$, we have $\sin \phi \cos \phi = \frac{1}{2} \sin \theta$ and $\sin^2 \phi = \frac{1}{2}(1 - \cos \theta)$, so

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

Along the cycloid, $\frac{dx}{d\theta} = a(1 - \cos \theta)$ and $\frac{dy}{d\theta} = a \sin \theta$, so, for $0 < \theta < 2\pi$,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin \theta}{1 - \cos \theta}.$$

Then

$$1 + \left(\frac{dy}{dx} \right)^2 = \frac{(1 - \cos \theta)^2 + \sin^2 \theta}{(1 - \cos \theta)^2} = \frac{2}{1 - \cos \theta},$$

and therefore $y(1 + y'^2) = a(1 - \cos \theta) \cdot \frac{2}{1 - \cos \theta} = 2a = k^2$.

d) Cycloid. We have

$$x'(\theta)^2 + y'(\theta)^2 = a^2 ((1 - \cos \theta)^2 + \sin^2 \theta) = 2a^2(1 - \cos \theta) = 4a^2 \sin^2(\theta/2),$$

so $\sqrt{x'(\theta)^2 + y'(\theta)^2} = 2a \sin(\theta/2)$ for $0 \leq \theta \leq \pi$. Also $\sqrt{y} = \sqrt{2a \sin^2(\theta/2)} = \sqrt{2a} \sin(\theta/2)$. Hence

$$J = \int_0^\pi \frac{2a \sin(\theta/2)}{\sqrt{2a} \sin(\theta/2)} d\theta = \int_0^\pi \sqrt{2a} d\theta = \pi \sqrt{2a}.$$

Straight line. The line from O to $P = (\pi a, 2a)$ is $y = \frac{2}{\pi}x$, so $\sqrt{1 + y'^2} = \frac{\sqrt{\pi^2 + 4}}{\pi}$ and $\frac{1}{\sqrt{y}} = \sqrt{\frac{\pi}{2}}x^{-1/2}$. Since $\int_0^{\pi a} x^{-1/2} dx = 2\sqrt{\pi a}$,

$$J = \frac{\sqrt{\pi^2 + 4}}{\pi} \sqrt{\frac{\pi}{2}} \cdot 2\sqrt{\pi a} = \sqrt{\pi^2 + 4} \sqrt{2a}.$$

Since $\pi < \sqrt{\pi^2 + 4}$, the cycloid is faster.

Problem 7

Consider the ODE

$$ty''(t) - (t+1)y'(t) + y(t) = 0.$$

- a) Verify that $y_1(t) = 1 + t$ and $y_2(t) = e^t$ are solutions for every $t \in \mathbb{R}$.
- b) Compute the Wronskian $W[y_1, y_2](t)$, and show that $\{y_1, y_2\}$ is a fundamental set of solutions on $I = (0, \infty)$.
- c) Write the ODE in the form $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on $I = (0, \infty)$, and check that your Wronskian satisfies Abel's identity.
- d) The Wronskian vanishes at $t = 0$ but at no other point. Explain why this does not contradict Corollary 3.1 in the notes, which states that the Wronskian of two solutions vanishes either everywhere or nowhere.
- e) Find the solution on $(0, \infty)$ that satisfies $y(1) = 0$ and $y'(1) = 1$.

Solutions: **a)** For $y_1 = 1 + t$, we have $y_1' = 1$ and $y_1'' = 0$, so $t \cdot 0 - (t+1) + (1+t) = 0$. For $y_2 = e^t$, we have $y_2' = y_2'' = e^t$, so $te^t - (t+1)e^t + e^t = 0$. Both hold for every $t \in \mathbb{R}$.

b) We have

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 = (1+t)e^t - e^t = te^t.$$

On $I = (0, \infty)$, the normalized equation of c) has continuous coefficients, and $W(t) = te^t \neq 0$. By Proposition 3.2 of the notes, y_1 and y_2 are linearly independent on I , so $\{y_1, y_2\}$ is a fundamental set of solutions on I .

c) Dividing by $t > 0$ gives

$$y''(t) - \left(1 + \frac{1}{t}\right)y'(t) + \frac{1}{t}y(t) = 0, \quad a_1(t) = -1 - \frac{1}{t}, \quad a_0(t) = \frac{1}{t}.$$

Abel's identity with $t_0 = 1$ predicts

$$W(t) = W(1)e^{-\int_1^t a_1(\tau) d\tau} = e \cdot e^{t-1+\ln t} = te^t,$$

in agreement with b).

d) Corollary 3.1 assumes that a_1 and a_0 are continuous on an open interval I . Here $a_1(t) = -1 - 1/t$ and $a_0(t) = 1/t$ are not defined at $t = 0$, so the corollary applies on $(0, \infty)$ and on $(-\infty, 0)$, but not on any interval containing 0. On each of these two intervals, $W(t) = te^t$ vanishes nowhere, as the corollary predicts.

e) By b), every solution on I has the form $y(t) = c_1(1+t) + c_2e^t$, so $y'(t) = c_1 + c_2e^t$. The initial conditions read

$$2c_1 + ec_2 = 0, \quad c_1 + ec_2 = 1.$$

Subtracting gives $c_1 = -1$, and then $c_2 = 2/e$. Therefore

$$y(t) = 2e^{t-1} - (1+t).$$

Problem 8

Let I be an open interval, and let a_1 and a_0 be continuous on I .

- a) Let y_1 and y_2 solve $y'' + a_1(t)y' + a_0(t)y = 0$ on I . Show that if y_1 and y_2 vanish at the same point $t_* \in I$, then $\{y_1, y_2\}$ is not a fundamental set of solutions.
- b) Show the same conclusion if y_1 and y_2 both have a local maximum at the same point $t_* \in I$.
- c) Conversely, let y_1 and y_2 be twice continuously differentiable functions on I whose Wronskian $W = W[y_1, y_2]$ does not vanish at any point of I . Show that y_1 and y_2 both solve exactly one equation of the form

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

with continuous coefficients on I , and that its coefficients are

$$a_1 = -\frac{W'}{W}, \quad a_0 = \frac{y_1'y_2'' - y_1''y_2'}{W}.$$

Hint: The two conditions $y_j'' + a_1y_j' + a_0y_j = 0$, $j = 1, 2$, form a linear system of two equations for the two unknowns $a_1(t)$ and $a_0(t)$.

- d) Apply c) to $y_1(t) = t$ and $y_2(t) = t^2$ on $I = (0, \infty)$.

Solutions: **a)** If $y_1(t_*) = y_2(t_*) = 0$, then $W(t_*) = y_1(t_*)y_2'(t_*) - y_1'(t_*)y_2(t_*) = 0$. By Proposition 3.2 of the notes, y_1 and y_2 are linearly dependent, so $\{y_1, y_2\}$ is not a fundamental set of solutions.

b) At an interior local maximum, the derivative vanishes, so $y_1'(t_*) = y_2'(t_*) = 0$. Then $W(t_*) = y_1(t_*) \cdot 0 - 0 \cdot y_2(t_*) = 0$, and we conclude as in a).

c) Fix $t \in I$. The conditions $y_j'' + a_1y_j' + a_0y_j = 0$, $j = 1, 2$, form the linear system

$$\begin{pmatrix} y_1' & y_1 \\ y_2' & y_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} -y_1'' \\ -y_2'' \end{pmatrix},$$

whose determinant $y_1'y_2 - y_1y_2' = -W$ is nonzero. Hence the system has exactly one solution, so at most one such equation exists. Cramer's rule gives

$$a_1 = \frac{-y_1''y_2 + y_1y_2''}{-W} = -\frac{y_1y_2'' - y_1''y_2}{W},$$

$$a_0 = \frac{-y_1'y_2'' + y_1''y_2'}{-W} = \frac{y_1'y_2'' - y_1''y_2'}{W}.$$

Differentiating $W = y_1y_2' - y_1'y_2$ gives $W' = y_1y_2'' - y_1''y_2$, so $a_1 = -W'/W$. Both coefficients are continuous on I because y_1 and y_2 are twice continuously differentiable and $W \neq 0$, and by construction y_1 and y_2 solve the equation with these coefficients.

d) With $y_1 = t$ and $y_2 = t^2$, we have $W = t \cdot 2t - 1 \cdot t^2 = t^2 \neq 0$ on $(0, \infty)$ and $W' = 2t$.
Then

$$a_1(t) = -\frac{2t}{t^2} = -\frac{2}{t}, \quad a_0(t) = \frac{1 \cdot 2 - 0 \cdot 2t}{t^2} = \frac{2}{t^2},$$

and the equation is

$$y''(t) - \frac{2}{t}y'(t) + \frac{2}{t^2}y(t) = 0, \quad t > 0,$$

or, after multiplication by t^2 , $t^2y''(t) - 2ty'(t) + 2y(t) = 0$.