

MATH 441: Mock in-class Quiz — Solutions

Problem 1 (12 points)

Consider the initial value problem

$$y'(t) = -2te^{-y(t)}, \quad y(0) = 0.$$

- a) Solve the problem explicitly for $y(t)$ using separation of variables.
- b) Give the largest open interval containing $t = 0$ on which your solution is defined, and compute the limit of $y(t)$ at each endpoint of that interval.

Solutions: **a)** The equation is separable, with $f(t) = -2t$ and $g(z) = e^{-z}$. The function g never vanishes, so we may divide by $g(y(t))$ and integrate from $t_0 = 0$:

$$\begin{aligned} e^{y(t)}y'(t) = -2t &\iff \int_{y(0)}^{y(t)} e^z dz = \int_0^t (-2\tau) d\tau \\ &\iff e^{y(t)} - e^0 = -t^2 \\ &\iff e^{y(t)} = 1 - t^2 \\ &\iff y(t) = \ln(1 - t^2). \end{aligned}$$

Let us check the answer. Differentiating gives $y'(t) = -2t/(1 - t^2)$, while $e^{-y(t)} = 1/(1 - t^2)$, so the ODE holds. In addition, $y(0) = \ln(1) = 0$.

b) The formula in a) makes sense exactly when $1 - t^2 > 0$, that is, when $-1 < t < 1$. The largest interval containing $t = 0$ on which the solution is defined is therefore $(-1, 1)$.

At either endpoint the argument of the logarithm tends to 0 from above, so

$$\lim_{t \rightarrow -1^+} y(t) = \lim_{t \rightarrow 1^-} y(t) = -\infty.$$

Observe that the right-hand side $(t, y) \mapsto -2te^{-y}$ is continuous on all of $\mathbb{R}^2$, yet the solution does not exist for all $t \in \mathbb{R}$.

Problem 2 (15 points)

Consider the initial value problem

$$y'(t) + ty(t) = t, \quad y(0) = 2.$$

- a) Solve the associated homogeneous ODE $y'(t) + ty(t) = 0$.
- b) Solve the initial value problem by variation of constants.
- c) Verify that the constant function $y_p \equiv 1$ solves the ODE. Use this observation together with your answer to a) to recover the answer to b) without computing an integral.

Solutions: **a)** The homogeneous ODE reads $y'(t) = a(t)y(t)$ with $a(t) = -t$, so $\int_0^t a(\tau) d\tau = -t^2/2$ and the homogeneous solutions are

$$y(t) = Le^{-t^2/2} \quad \text{for some constant } L.$$

b) Following the method of variation of constants, we replace the constant L by a function and set $y(t) = L(t)R(t)$ with $R(t) = e^{-t^2/2}$. Since $R'(t) = -tR(t)$, the product rule gives

$$y'(t) = L'(t)R(t) + L(t)R'(t) = L'(t)e^{-t^2/2} - ty(t),$$

and therefore $y'(t) + ty(t) = L'(t)e^{-t^2/2}$. Comparing with the ODE yields

$$L'(t)e^{-t^2/2} = t \iff L'(t) = te^{t^2/2} \implies L(t) = e^{t^2/2} + C$$

for some constant C . Consequently,

$$y(t) = \left(e^{t^2/2} + C\right)e^{-t^2/2} = 1 + Ce^{-t^2/2}.$$

The initial condition $y(0) = 2$ gives $1 + C = 2$, that is, $C = 1$. We conclude

$$y(t) = 1 + e^{-t^2/2}.$$

Let us check the answer. Differentiating gives $y'(t) = -te^{-t^2/2}$, so $y'(t) + ty(t) = -te^{-t^2/2} + t + te^{-t^2/2} = t$, and $y(0) = 2$.

c) The function $y_p \equiv 1$ is constant, so $y'_p(t) = 0$, and

$$y'_p(t) + ty_p(t) = 0 + t \cdot 1 = t,$$

so y_p solves the ODE. The ODE is linear, so the difference of any two of its solutions solves the homogeneous ODE. Hence, if y is any solution, then $y - y_p$ is one of the functions found in a), and

$$y(t) = 1 + Le^{-t^2/2} \quad \text{for some constant } L.$$

Imposing $y(0) = 2$ gives $L = 1$ and returns the answer of b), with no integration required.

Problem 3 (13 points)

Consider the autonomous ODE

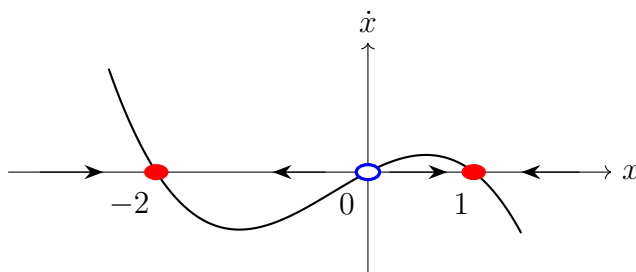
$$\dot{x} = x(1-x)(x+2).$$

- Find all fixed points and sketch the graph of $\dot{x}$ against x , marking the direction of the flow on the x -axis.
- Classify each fixed point as stable or unstable.
- Determine $\lim_{t \rightarrow \infty} x(t)$ for each of the initial values $x(0) = -1$, $x(0) = 1/2$ and $x(0) = 3$, without solving the ODE.

Solutions: **a)** The fixed points are the roots of $x(1-x)(x+2)$, namely $x = -2$, $x = 0$ and $x = 1$. The sign of $\dot{x}$ on each of the four intervals between consecutive roots follows from the signs of the three factors:

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
x	—	—	+	+
$1 - x$	+	+	+	—
$x + 2$	—	+	+	+
$\dot{x}$	+	—	+	—

Hence the flow points to the right for $x < -2$ and for $0 < x < 1$, and to the left for $-2 < x < 0$ and for $x > 1$.



b) The fixed points $x = -2$ and $x = 1$ are stable: at each of them the flow points to the right immediately to its left and to the left immediately to its right, so nearby solutions converge to it.

The fixed point $x = 0$ is unstable: the flow points to the left immediately to its left and to the right immediately to its right, so solutions starting arbitrarily close to 0 move away from it.

c) Write $f(x) = x(1-x)(x+2)$. The function f is continuously differentiable, so solutions are unique by the Picard–Lindelöf theorem, and a solution that reaches a fixed point coincides with the constant solution there. Hence a solution cannot cross a fixed point, and it is monotone on each interval between consecutive fixed points, where f has a constant sign.

Moreover, if $x(t)$ converges to a limit x^* as $t \rightarrow \infty$, then x^* is a fixed point. Indeed, $\dot{x}(t) = f(x(t)) \rightarrow f(x^*)$ by continuity of f , and $f(x^*) \neq 0$ would force $|x(t)| \rightarrow \infty$. We read the three limits off the phase line:

- For $x(0) = -1 \in (-2, 0)$, the flow points to the left, so x decreases and stays above -2 . Hence $x(t)$ converges to a fixed point in $[-2, -1)$, and the only one is -2 . We conclude $\lim_{t \rightarrow \infty} x(t) = -2$.
- For $x(0) = 1/2 \in (0, 1)$, the flow points to the right, so x increases and stays below 1 . Hence $x(t)$ converges to a fixed point in $(1/2, 1]$, and the only one is 1 . We conclude $\lim_{t \rightarrow \infty} x(t) = 1$.
- For $x(0) = 3 > 1$, the flow points to the left, so x decreases and stays above 1 . Hence $x(t)$ converges to a fixed point in $[1, 3)$, and the only one is 1 . We conclude $\lim_{t \rightarrow \infty} x(t) = 1$.

Problem 4 (10 points)

Consider the initial value problem

$$y'(t) = t + \sin(y(t)), \quad y(0) = 0.$$

Fix $T > 0$ and let $D_T = [0, T] \times [-C, C]$, where $C > 0$ is to be chosen.

- Verify assumptions (A1), (A2) and (A3) of the Picard–Lindelöf theorem on D_T , and give explicit values of the constants K and L .
- Choose C so that the theorem applies on $[0, T]$. Conclude that the initial value problem has exactly one continuously differentiable solution on $[0, \infty)$.

Solutions: a) Write $f(t, y) = t + \sin(y)$, so that $t_0 = 0$ and $y_0 = 0$.

(A1) The function f is a sum of a polynomial in t and a continuous function of y , hence f is continuous on D_T .

(A2) Since $y_0 = 0$, we have $|f(t, y_0)| = |t + \sin(0)| = t$ for every $t \in [0, T]$, and therefore

$$|f(t, y_0)| \leq T =: K \quad \forall t \in [0, T].$$

(A3) Let $t \in [0, T]$ and let $y_1, y_2 \in [-C, C]$. The mean value theorem applied to the sine yields a point ξ between y_1 and y_2 such that $\sin(y_1) - \sin(y_2) = \cos(\xi)(y_1 - y_2)$. Since $|\cos(\xi)| \leq 1$, we conclude

$$|f(t, y_1) - f(t, y_2)| = |\sin(y_1) - \sin(y_2)| \leq |y_1 - y_2|,$$

so that f is Lipschitz continuous in y on D_T with constant $L := 1$. Note that the t term cancels; it plays no role in (A3).

b) The hypothesis of the Picard–Lindelöf theorem requires

$$C \geq \frac{K}{L} (e^{L(T-t_0)} - 1) = T (e^T - 1).$$

The right-hand side is a finite positive number, so the choice

$$C := T(e^T - 1)$$

is admissible. The theorem then gives a unique continuously differentiable solution y_T on $[0, T]$, whose graph lies in D_T .

Since $T > 0$ was arbitrary, the solution y_T exists for every $T > 0$. Let $0 < T < T'$. The restriction $y_{T'}|_{[0, T]}$ solves the initial value problem on $[0, T]$, so $y_{T'} = y_T$ on $[0, T]$ by the uniqueness statement on $[0, T]$. For $t \geq 0$ we may therefore define $y(t) := y_T(t)$ for any $T > t$, and the value does not depend on the choice of T . The function y is continuously differentiable and solves the initial value problem on $[0, \infty)$.

For uniqueness, suppose u_1 and u_2 both solve the initial value problem on $[0, \infty)$ and let $t \geq 0$. Pick $T > t$. Both restrictions $u_1|_{[0, T]}$ and $u_2|_{[0, T]}$ solve the initial value problem on $[0, T]$, so they agree by the uniqueness statement above, and in particular $u_1(t) = u_2(t)$. Hence $u_1 = u_2$ on $[0, \infty)$.