

MATH 441: (Practice) Mock in-class Quiz — Solutions

Problem 1 (12 points)

Consider the initial value problem

$$y'(t) = (1 - 2t)e^{-y(t)}, \quad y(0) = 0.$$

- a) Solve the problem explicitly for $y(t)$ using separation of variables.
- b) Give the largest open interval containing $t = 0$ on which your solution is defined, and compute the limit of $y(t)$ at each endpoint of that interval.

Solutions: a) The equation is separable, with $f(t) = 1 - 2t$ and $g(z) = e^{-z}$. The function g never vanishes, so we may divide by $g(y(t))$ and integrate from $t_0 = 0$:

$$\begin{aligned} e^{y(t)} y'(t) = 1 - 2t &\iff \int_{y(0)}^{y(t)} e^z dz = \int_0^t (1 - 2\tau) d\tau \\ &\iff e^{y(t)} - e^0 = t - t^2 \\ &\iff e^{y(t)} = 1 + t - t^2 \\ &\iff y(t) = \ln(1 + t - t^2). \end{aligned}$$

Let us check the answer. Differentiating gives $y'(t) = (1 - 2t)/(1 + t - t^2)$, while $e^{-y(t)} = 1/(1 + t - t^2)$, so the ODE holds. In addition, $y(0) = \ln(1) = 0$.

b) The formula in a) makes sense exactly when $1 + t - t^2 > 0$, that is, when $t^2 - t - 1 < 0$. The quadratic $t^2 - t - 1$ has the two real roots

$$t_- = \frac{1 - \sqrt{5}}{2} \approx -0.618 \quad \text{and} \quad t_+ = \frac{1 + \sqrt{5}}{2} \approx 1.618,$$

and it is negative between them. The largest interval containing $t = 0$ on which the solution is defined is therefore (t_-, t_+) .

At either endpoint the argument of the logarithm tends to 0 from above, so

$$\lim_{t \rightarrow t_-^+} y(t) = \lim_{t \rightarrow t_+^-} y(t) = -\infty.$$

Observe that the right-hand side $(t, y) \mapsto (1 - 2t)e^{-y}$ is continuous on all of $\mathbb{R}^2$, yet the solution does not exist for all $t \in \mathbb{R}$.

Problem 2 (15 points)

Consider the initial value problem

$$y'(t) - 2ty(t) = t, \quad y(0) = 1.$$

- a) Solve the associated homogeneous ODE $y'(t) - 2ty(t) = 0$.
- b) Solve the initial value problem by variation of constants.
- c) Verify that the constant function $y_p \equiv -1/2$ solves the ODE. Use this observation together with your answer to a) to recover the answer to b) without computing an integral.

Solutions: **a)** Here $a(t) = 2t$, so $\int_0^t a(\tau) d\tau = t^2$ and the homogeneous solutions are

$$y(t) = Le^{t^2} \quad \text{for some constant } L.$$

b) Following the method of variation of constants, we replace the constant L by a function and set $y(t) = L(t)R(t)$ with $R(t) = e^{t^2}$. Since $R'(t) = 2tR(t)$, the product rule gives

$$y'(t) = L'(t)R(t) + L(t)R'(t) = L'(t)e^{t^2} + 2ty(t),$$

and therefore $y'(t) - 2ty(t) = L'(t)e^{t^2}$. Comparing with the ODE yields

$$L'(t)e^{t^2} = t \iff L'(t) = te^{-t^2} \implies L(t) = -\frac{1}{2}e^{-t^2} + C$$

for some constant C . Consequently,

$$y(t) = \left(-\frac{1}{2}e^{-t^2} + C\right)e^{t^2} = -\frac{1}{2} + Ce^{t^2}.$$

The initial condition $y(0) = 1$ gives $-1/2 + C = 1$, that is, $C = 3/2$. We conclude

$$y(t) = -\frac{1}{2} + \frac{3}{2}e^{t^2}.$$

c) The function $y_p \equiv -1/2$ is constant, so $y'_p(t) = 0$, and

$$y'_p(t) - 2ty_p(t) = 0 - 2t\left(-\frac{1}{2}\right) = t,$$

so y_p solves the ODE. The ODE is linear, so the difference of any two of its solutions solves the homogeneous ODE. Hence, if y is any solution, then $y - y_p$ is one of the functions found in a), and

$$y(t) = -\frac{1}{2} + Le^{t^2} \quad \text{for some constant } L.$$

Imposing $y(0) = 1$ gives $L = 3/2$ and returns the answer of b), with no integration required.

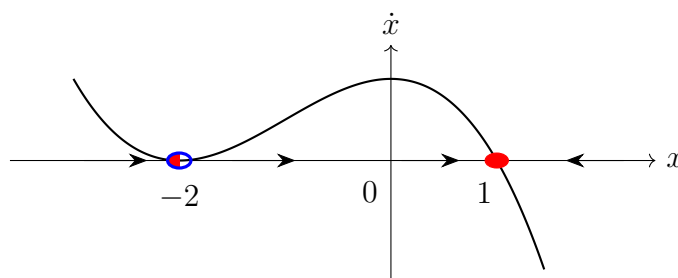
Problem 3 (13 points)

Consider the autonomous ODE

$$\dot{x} = (1 - x)(x + 2)^2.$$

- Find all fixed points and sketch the graph of $\dot{x}$ against x , marking the direction of the flow on the x -axis.
- Classify each fixed point as stable or unstable.
- Determine $\lim_{t \rightarrow \infty} x(t)$ for each of the initial values $x(0) = -3$, $x(0) = 0$ and $x(0) = 4$, without solving the ODE.

Solutions: **a)** The fixed points are the roots of $(1 - x)(x + 2)^2$, namely $x = 1$ and $x = -2$. The factor $(x + 2)^2$ is nonnegative and vanishes only at $x = -2$, so the sign of $\dot{x}$ is the sign of $1 - x$ away from $x = -2$. Hence $\dot{x} > 0$ for $x < 1$ with $x \neq -2$, and $\dot{x} < 0$ for $x > 1$: the flow points to the right everywhere left of $x = 1$, and to the left everywhere right of $x = 1$.



b) The fixed point $x = 1$ is stable: the flow points to the right immediately to its left and to the left immediately to its right, so nearby solutions converge to it.

The fixed point $x = -2$ is unstable. Indeed, the flow points to the right on both sides of $x = -2$, so solutions starting slightly below -2 converge to -2 , whereas solutions starting slightly above -2 move away from it. Some solutions starting arbitrarily close to -2 therefore leave every small neighborhood of -2 , even though the fixed point attracts solutions from the left. The half-filled dot in the sketch records this one-sided behavior.

c) Write $f(x) = (1 - x)(x + 2)^2$. The function f is continuously differentiable, so solutions are unique by the Picard–Lindelöf theorem, and a solution that reaches a fixed point coincides with the constant solution there. Hence a solution cannot cross a fixed point, and it is monotone on each interval between consecutive fixed points, where f has a constant sign. Moreover, if $x(t)$ converges to a limit x^* as $t \rightarrow \infty$, then x^* is a fixed point. Indeed, $\dot{x}(t) = f(x(t)) \rightarrow f(x^*)$ by continuity of f , and $f(x^*) \neq 0$ would force $|x(t)| \rightarrow \infty$. We read the three limits off the phase line:

- For $x(0) = -3 < -2$, the flow points to the right, so x increases and stays below -2 . Hence $x(t)$ converges to a fixed point in $(-3, -2]$, and the only one is -2 . We conclude $\lim_{t \rightarrow \infty} x(t) = -2$.

- For $x(0) = 0 \in (-2, 1)$, the flow points to the right, so x increases and stays below 1. Hence $x(t)$ converges to a fixed point in $(0, 1]$, and the only one is 1. We conclude $\lim_{t \rightarrow \infty} x(t) = 1$.
- For $x(0) = 4 > 1$, the flow points to the left, so x decreases and stays above 1. Hence $x(t)$ converges to a fixed point in $[1, 4)$, and the only one is 1. We conclude $\lim_{t \rightarrow \infty} x(t) = 1$.

Problem 4 (10 points)

Consider the initial value problem

$$y'(t) = t^2 + \cos(y(t)), \quad y(0) = 0.$$

Fix $T > 0$ and let $D_T = [0, T] \times [-C, C]$, where $C > 0$ is to be chosen.

- Verify assumptions (A1), (A2) and (A3) of the Picard–Lindelöf theorem on D_T , and give explicit values of the constants K and L .
- Choose C so that the theorem applies on $[0, T]$. Conclude that the initial value problem has exactly one continuously differentiable solution on $[0, \infty)$.

Solutions: a) Write $f(t, y) = t^2 + \cos(y)$, so that $t_0 = 0$ and $y_0 = 0$.

(A1) The function f is a sum of a polynomial in t and a continuous function of y , hence f is continuous on D_T .

(A2) Since $y_0 = 0$, we have $|f(t, y_0)| = |t^2 + \cos(0)| = t^2 + 1$ for every $t \in [0, T]$, and therefore

$$|f(t, y_0)| \leq T^2 + 1 =: K \quad \forall t \in [0, T].$$

(A3) Let $t \in [0, T]$ and let $y_1, y_2 \in [-C, C]$. The mean value theorem applied to the cosine yields a point ξ between y_1 and y_2 such that $\cos(y_1) - \cos(y_2) = -\sin(\xi)(y_1 - y_2)$. Since $|\sin(\xi)| \leq 1$, we conclude

$$|f(t, y_1) - f(t, y_2)| = |\cos(y_1) - \cos(y_2)| \leq |y_1 - y_2|,$$

so that f is Lipschitz continuous in y on D_T with constant $L := 1$. Note that the t^2 term cancels; it plays no role in (A3).

b) The hypothesis of the Picard–Lindelöf theorem requires

$$C \geq \frac{K}{L} (e^{L(T-t_0)} - 1) = (T^2 + 1) (e^T - 1).$$

The right-hand side is a finite positive number, so the choice

$$C := (T^2 + 1) (e^T - 1)$$

is admissible. The theorem then gives a unique continuously differentiable solution y_T on $[0, T]$, whose graph lies in D_T .

Since $T > 0$ was arbitrary, the solution y_T exists for every $T > 0$. Let $0 < T < T'$. The restriction $y_{T'}|_{[0,T]}$ solves the initial value problem on $[0, T]$, so $y_{T'} = y_T$ on $[0, T]$ by the uniqueness statement on $[0, T]$. For $t \geq 0$ we may therefore define $y(t) := y_T(t)$ for any $T > t$, and the value does not depend on the choice of T . The function y is continuously differentiable and solves the initial value problem on $[0, \infty)$.

For uniqueness, suppose u_1 and u_2 both solve the initial value problem on $[0, \infty)$ and let $t \geq 0$. Pick $T > t$. Both restrictions $u_1|_{[0,T]}$ and $u_2|_{[0,T]}$ solve the initial value problem on $[0, T]$, so they agree by the uniqueness statement above, and in particular $u_1(t) = u_2(t)$. Hence $u_1 = u_2$ on $[0, \infty)$.