3 Second-order linear ODEs

We now turn to the second-order linear ODE

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t),$$

where the coefficients a_1 and a_0 and the inhomogeneity f are given functions. This chapter develops the structure of the set of solutions (superposition, the Wronskian, and Abel's identity), solves equations with constant coefficients through the characteristic polynomial, and constructs particular solutions by undetermined coefficients and by variation of constants. We close the chapter with the damped oscillation, forced vibrations, and resonance.

Example 3.1 (The linearized pendulum, revisited). As we saw in Example 1.1, small oscillations of a pendulum of length ℓ obey the harmonic oscillator

$$\theta''(t) + \omega_0^2\theta(t) = 0, \qquad \omega_0 = \sqrt{g/\ell}.$$

We claimed there that the general solution is a combination of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$. Both functions solve the ODE, since $\frac{d^2}{dt^2}\cos(\omega_0 t) = -\omega_0^2\cos(\omega_0 t)$ and $\frac{d^2}{dt^2}\sin(\omega_0 t) = -\omega_0^2\sin(\omega_0 t)$, and so does every combination

$$\theta(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t), \qquad c_1, c_2 \in \mathbb{R}.$$

Suppose we release the pendulum at the angle θ_0 with angular velocity v_0, that is, $\theta(0) = \theta_0$ and $\theta'(0) = v_0$. Then $\theta(0) = c_1$ and $\theta'(0) = \omega_0 c_2$, so the choice $c_1 = \theta_0$ and $c_2 = v_0/\omega_0$ matches both initial conditions:

$$\theta(t) = \theta_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

To read off the amplitude, set $A = \sqrt{\theta_0^2 + (v_0/\omega_0)^2}$ and suppose that $A > 0$ (otherwise $\theta \equiv 0$). The point $(\theta_0/A, v_0/(\omega_0 A))$ lies on the unit circle, so there is an angle $\varphi \in [0, 2\pi)$ with $\cos\varphi = \theta_0/A$ and $\sin\varphi = v_0/(\omega_0 A)$. The addition formula $\cos(a-b) = \cos a\cos b + \sin a\sin b$ then gives

$$\theta(t) = A\left(\cos\varphi\cos(\omega_0 t) + \sin\varphi\sin(\omega_0 t)\right) = A\cos(\omega_0 t - \varphi).$$

The pendulum oscillates with amplitude A and period $2\pi/\omega_0 = 2\pi\sqrt{\ell/g}$. In this linear model, the period depends on the length of the pendulum but not on the amplitude. For example, a pendulum of length $\ell = 1$ m swings with period $2\pi/\sqrt{9.81} \approx 2.01$ s, whether we release it from rest at $\theta_0 = 0.05$ or at $\theta_0 = 0.1$ (in radians).

1) Are there other solutions?

2) Where do $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ come from?

3) What happens with friction, or with an external force pushing the pendulum?

3.1 Linear theory

Let $I \subseteq \mathbb{R}$ denote an open interval and let the functions $a_1, a_0, f \colon I \to \mathbb{R}$ be continuous. We consider the equation

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t), \qquad t \in I, \tag{7}$$

together with its associated homogeneous equation

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0, \qquad t \in I. \tag{8}$$

A *solution* is a twice differentiable function $y \colon I \to \mathbb{R}$ that satisfies the equation at every $t \in I$. As we saw in Section 1.3, an nth-order ODE typically requires n additional conditions. For an IVP, we prescribe the solution and its derivative at a point $t_0 \in I$:

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{9}$$

where y_0 and y_0' are given real numbers.

Theorem 3.1 (Existence and uniqueness for linear equations)**.** *Let a_1, a_0, and f be continuous on the open interval I, let $t_0 \in I$, and let $y_0, y_0' \in \mathbb{R}$. Then the initial value problem* (7)–(9) *has exactly one solution on I.*

Proof. Deferred to later in the course. □

Proposition 3.1 (Superposition principle)**.** *If y_1 and y_2 solve the homogeneous equation* (8) *and $c_1, c_2 \in \mathbb{R}$, then $c_1y_1 + c_2y_2$ also solves* (8).

Proof. Differentiation is linear, so for every $t \in I$,

$$\begin{aligned}&(c_1y_1 + c_2y_2)''(t) + a_1(t)(c_1y_1 + c_2y_2)'(t) + a_0(t)(c_1y_1 + c_2y_2)(t)\\ &\qquad = c_1\left(y_1''(t) + a_1(t)y_1'(t) + a_0(t)y_1(t)\right) + c_2\left(y_2''(t) + a_1(t)y_2'(t) + a_0(t)y_2(t)\right) = 0.\end{aligned}$$

□

Proposition 3.1 shows the solutions of (8) form a vector space. How large is this space?

3.1.1 Linear independence of functions

Definition 3.1. Let $g_1, \ldots, g_n \colon I \to \mathbb{R}$ be functions.

a) An expression of the form $\alpha_1g_1 + \cdots + \alpha_ng_n$, with $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, is called a *linear combination* of $g_1, \ldots, g_n$.

b) The set of all linear combinations of $g_1, \ldots, g_n$ is called the *span* of $\{g_1, \ldots, g_n\}$ and is denoted by

$$\operatorname{Span}(\{g_1, \ldots, g_n\}) := \{\alpha_1g_1 + \cdots + \alpha_ng_n : \alpha_1, \ldots, \alpha_n \in \mathbb{R}\}.$$

c) The functions $g_1, \ldots, g_n$ are *linearly independent* if none of them can be written as a linear combination of the others. Otherwise, they are *linearly dependent*.

Lemma 3.1. *The functions $g_1, \dots, g_n \colon I \to \mathbb{R}$ are linearly independent if and only if the identity*

$$\alpha_1 g_1(t) + \alpha_2 g_2(t) + \cdots + \alpha_n g_n(t) = 0 \quad \text{for every } t \in I \tag{$*$}$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Proof. "$\Longrightarrow$" (only if): Suppose that $g_1, \dots, g_n$ are linearly independent, and suppose, for contradiction, that $(*)$ holds with $\alpha_j \neq 0$ for some j. Dividing $(*)$ by α_j gives

$$g_j(t) = -\sum_{i \neq j} \frac{\alpha_i}{\alpha_j} g_i(t) \quad \text{for every } t \in I,$$

so g_j is a linear combination of the others, a contradiction.

"$\Longleftarrow$" (if): Suppose that $(*)$ forces $\alpha_1 = \cdots = \alpha_n = 0$. If some g_j satisfied $g_j = \sum_{i \neq j} \beta_i g_i$, then $\sum_{i \neq j} \beta_i g_i - g_j = 0$ on I would be an identity of the form $(*)$ in which the coefficient of g_j equals $-1 \neq 0$, a contradiction. Hence $g_1, \dots, g_n$ are linearly independent. $\square$

Example 3.2. a) Are $g_1(t) = t^2$ and $g_2(t) = t + 1$ linearly independent on $\mathbb{R}$? Suppose that $\alpha_1 t^2 + \alpha_2(t + 1) = 0$ for every $t \in \mathbb{R}$. Differentiating twice gives $2\alpha_1 = 0$, and differentiating once and setting $\alpha_1 = 0$ gives $\alpha_2 = 0$. By Lemma 3.1, the functions are linearly independent.

b) Are $g_1(t) = 1 + t$, $g_2(t) = 1 - t$, and $g_3(t) = 2t$ linearly independent on $\mathbb{R}$? The identity $\alpha_1(1 + t) + \alpha_2(1 - t) + \alpha_3(2t) = 0$ for every $t \in \mathbb{R}$ is equivalent to

$$(\alpha_1 + \alpha_2) + (\alpha_1 - \alpha_2 + 2\alpha_3)t = 0 \quad \text{for every } t \in \mathbb{R}.$$

Setting $t = 0$ and then differentiating shows that both coefficients vanish: $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 + 2\alpha_3 = 0$, that is, $\alpha_2 = -\alpha_1$ and $\alpha_3 = -\alpha_1$. Any choice $\alpha_1 \neq 0$ shows linear dependence; for example, $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = -1$ give

$$(1 + t) - (1 - t) - 2t = 0. \checkmark$$

Definition 3.2. Let V be a vector space of functions on I (for example, a span). A set $\{g_1, \dots, g_k\} \subseteq V$ is a *basis* of V if $g_1, \dots, g_k$ are linearly independent and $\operatorname{Span}(\{g_1, \dots, g_k\}) = V$. All bases of V have the same number of elements (a fact from linear algebra); this number is called the *dimension* of V and is denoted by $\dim V$.

Example 3.3. Let $S = \{1 + t, 1 - t, 2t, -5\}$, and label these functions g_1, g_2, g_3, g_4. What is $\dim(\operatorname{Span}(S))$? Note that $g_1 = \frac{g_3}{2} - \frac{g_4}{5}$ and $g_2 = -\frac{g_3}{2} - \frac{g_4}{5}$, so $\operatorname{Span}(S) = \operatorname{Span}(\{1, t\})$. The functions 1 and t are linearly independent, hence $\dim(\operatorname{Span}(S)) = 2$, and a basis is $\{1, t\}$.

3.1.2 The Wronskian

Let y_1 and y_2 solve (8). Can we fit the combination $c_1 y_1 + c_2 y_2$, which solves (8) by superposition, to the initial conditions (9)? The conditions read $c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$ and $c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$, or, in matrix form,

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}. \tag{10}$$

This linear system has a unique solution for every choice of y_0 and y_0' if and only if the determinant of its matrix is nonzero.

Definition 3.3. The *Wronskian* of two differentiable functions $y_1, y_2 \colon I \to \mathbb{R}$ is the function

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t), \qquad t \in I.$$

When the functions y_1 and y_2 are clear from the context, we simply write $W(t)$.

Proposition 3.2. *Let y_1 and y_2 solve* (8) *on I. Then y_1 and y_2 are linearly independent if and only if $W(t) \neq 0$ for every $t \in I$.*

Proof. "$\Longleftarrow$" (if): Suppose $W(t) \neq 0$ for every $t \in I$ and suppose $c_1y_1(t) + c_2y_2(t) = 0$ for every $t \in I$. Differentiating this identity gives $c_1y_1'(t) + c_2y_2'(t) = 0$ for every $t \in I$, that is,

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for every } t \in I.$$

The determinant of this matrix is $W(t) \neq 0$, so $c_1 = c_2 = 0$. By Lemma 3.1, the functions y_1 and y_2 are linearly independent.

"$\Longrightarrow$" (only if): Suppose that y_1 and y_2 are linearly independent and that $W(t_*) = 0$ for some $t_* \in I$. Then the matrix with columns $(y_1(t_*), y_1'(t_*))$ and $(y_2(t_*), y_2'(t_*))$ is singular, so there exist $\tilde{c}_1, \tilde{c}_2$, not both zero, such that

$$\begin{pmatrix} y_1(t_*) & y_2(t_*) \\ y_1'(t_*) & y_2'(t_*) \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider the function $y = \tilde{c}_1y_1 + \tilde{c}_2y_2$. It solves (8) by Proposition 3.1, and it satisfies $y(t_*) = 0$ and $y'(t_*) = 0$. The zero function solves the same initial value problem, so the uniqueness part of Theorem 3.1 forces $y \equiv 0$ on I. Thus $\tilde{c}_1y_1 + \tilde{c}_2y_2 = 0$ on I with $\tilde{c}_1, \tilde{c}_2$ not both zero, and y_1 and y_2 are linearly dependent, which contradicts our assumption. ☐

Proposition 3.3 (Abel's identity)**.** *Let y_1 and y_2 solve* (8) *on I, and let $t_0 \in I$. Then*

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)\,d\tau} \qquad \text{for every } t \in I.$$

This result is also known as the *Abel–Jacobi–Liouville identity* (or *Liouville's formula*).

Proof. Differentiating $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ gives

$$W'(t) = y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_1''(t)y_2(t) - y_1'(t)y_2'(t) = y_1(t)y_2''(t) - y_1''(t)y_2(t).$$

Now replace y_1'' and y_2'' using the ODE: $y_j''(t) = -a_1(t)y_j'(t) - a_0(t)y_j(t)$ for $j = 1, 2$. Then

$$\begin{aligned} W'(t) &= -a_1(t)y_1(t)y_2'(t) - a_0(t)y_1(t)y_2(t) + a_1(t)y_1'(t)y_2(t) + a_0(t)y_1(t)y_2(t) \\ &= -a_1(t)\left(y_1(t)y_2'(t) - y_1'(t)y_2(t)\right) = -a_1(t)W(t). \end{aligned}$$

Hence W solves the first-order linear homogeneous ODE $W'(t) = -a_1(t)W(t)$. We showed in Section 2.1 that every solution of this equation has the form $W(t) = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)\,d\tau}$. (Note that this argument does not require $W(t) \neq 0$.) ☐

Corollary 3.1. *Let y_1 and y_2 solve* (8) *on I. Then either $W(t) = 0$ for every $t \in I$, or $W(t) \neq 0$ for every $t \in I$. In particular, y_1 and y_2 are linearly independent if and only if $W(t_*) \neq 0$ for some $t_* \in I$.*

Proof. The exponential factor in Proposition 3.3 is positive, so $W(t) = 0$ if and only if $W(t_0) = 0$, and $t_0 \in I$ is arbitrary. The second statement then follows from Proposition 3.2. $\square$

Corollary 3.1 gives a practical test: to decide whether two solutions are linearly independent, we compute their Wronskian at a single, convenient point.

Example 3.4. Consider the ODE $y''(t) - y(t) = 0$ on $I = \mathbb{R}$. Direct differentiation shows that e^t, $\cosh t = \frac{1}{2}(e^t + e^{-t})$, and $\sinh t = \frac{1}{2}(e^t - e^{-t})$ solve it, since $\cosh' = \sinh$ and $\sinh' = \cosh$. All three functions and their derivatives take simple values at $t = 0$, so we evaluate the Wronskian there.

a) Are $y_1(t) = \cosh t$ and $y_2(t) = \sinh t$ linearly independent? We have

$$W(0) = \cosh(0)\cosh(0) - \sinh(0)\sinh(0) = 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0,$$

so Corollary 3.1 shows that y_1 and y_2 are linearly independent.

b) Are $y_1(t) = e^t$ and $y_2(t) = \cosh t + \sinh t$ linearly independent? Here $y_1(0) = y_1'(0) = 1$ and $y_2(0) = y_2'(0) = 1$, so

$$W(0) = 1 \cdot 1 - 1 \cdot 1 = 0.$$

By Corollary 3.1, the functions y_1 and y_2 are linearly dependent. Indeed, $\cosh t + \sinh t = e^t$ for every $t \in \mathbb{R}$.

Theorem 3.2 (Structure of the solution set)**.** *(a) The solutions of the homogeneous equation* (8) *form a vector space of dimension two.*

(b) If y_1 and y_2 are linearly independent solutions of (8)*, then every solution y of* (8) *can be written as $y = c_1 y_1 + c_2 y_2$ for unique constants $c_1, c_2 \in \mathbb{R}$.*

Proof. (a) By Proposition 3.1, the solutions of (8) form a vector space, which we denote by S. Fix $t_0 \in I$. By Theorem 3.1, there exist solutions $z_1, z_2 \in S$ with

$$z_1(t_0) = 1, \quad z_1'(t_0) = 0, \qquad z_2(t_0) = 0, \quad z_2'(t_0) = 1.$$

Their Wronskian at t_0 equals $1 \cdot 1 - 0 \cdot 0 = 1$, so z_1 and z_2 are linearly independent by Corollary 3.1. They also span S. Indeed, given $y \in S$, the function $w = y - y(t_0)z_1 - y'(t_0)z_2$ lies in S and satisfies $w(t_0) = 0$ and $w'(t_0) = 0$. The zero function solves the same initial value problem, so $w \equiv 0$ by the uniqueness part of Theorem 3.1, and $y = y(t_0)z_1 + y'(t_0)z_2$. Hence $\{z_1, z_2\}$ is a basis of S, and $\dim S = 2$.

(b) Let $y \in S$. By Corollary 3.1, $W(t_0) \neq 0$, so the system (10) with $y_0 = y(t_0)$ and $y_0' = y'(t_0)$ has a unique solution (c_1, c_2). The functions y and $c_1 y_1 + c_2 y_2$ then solve the same initial value problem, and the uniqueness part of Theorem 3.1 gives $y = c_1 y_1 + c_2 y_2$. The constants are unique because y_1 and y_2 are linearly independent. $\square$

Definition 3.4. A pair $\{y_1, y_2\}$ of linearly independent solutions of (8) is called a *fundamental set of solutions*, and the expression $c_1y_1 + c_2y_2$ with arbitrary constants $c_1, c_2 \in \mathbb{R}$ is called the *general solution* of (8).

To solve an initial value problem, we find the constants from the system (10).

Example 3.5. Consider the ODE $2t^2y''(t) + 3ty'(t) - y(t) = 0$ for $t > 0$. Dividing by $2t^2$, we see that this equation is equivalent to

$$y''(t) + \frac{3}{2t}y'(t) - \frac{1}{2t^2}y(t) = 0,$$

whose coefficients are continuous on $I = (0, \infty)$, so the theory above applies on I. We claim that $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ form a fundamental set of solutions. Indeed, we have $y_1'(t) = \frac{1}{2\sqrt{t}}$, $y_1''(t) = -\frac{1}{4t^{3/2}}$, $y_2'(t) = -\frac{1}{t^2}$, and $y_2''(t) = \frac{2}{t^3}$, so

$$2t^2\left(-\frac{1}{4t^{3/2}}\right) + 3t \cdot \frac{1}{2\sqrt{t}} - \sqrt{t} = -\frac{1}{2}\sqrt{t} + \frac{3}{2}\sqrt{t} - \sqrt{t} = 0$$

and

$$2t^2 \cdot \frac{2}{t^3} + 3t\left(-\frac{1}{t^2}\right) - \frac{1}{t} = \frac{4}{t} - \frac{3}{t} - \frac{1}{t} = 0.$$

Are y_1 and y_2 linearly independent? Compute the Wronskian:

$$W(t) = \begin{vmatrix} \sqrt{t} & 1/t \\ \frac{1}{2\sqrt{t}} & -1/t^2 \end{vmatrix} = \sqrt{t}\left(-\frac{1}{t^2}\right) - \frac{1}{2\sqrt{t}} \cdot \frac{1}{t} = -\frac{3}{2t^{3/2}} \neq 0 \quad \forall\, t > 0.$$

Therefore, $\sqrt{t}$ and $1/t$ form a fundamental set of solutions, and the general solution of the ODE is $y(t) = c_1\sqrt{t} + c_2/t$. As a check, Abel's identity with $a_1(t) = \frac{3}{2t}$ and $t_0 = 1$ predicts $W(t) = W(1)e^{-\int_1^t \frac{3}{2\tau}\,d\tau} = -\frac{3}{2}t^{-3/2}$, in agreement with the direct computation.

3.1.3 Inhomogeneous equations

Warning: when $f \not\equiv 0$, the solutions of (7) do *not* form a vector space! However, suppose y and y_p both solve (7). Subtracting the two equations gives

$$(y - y_p)''(t) + a_1(t)(y - y_p)'(t) + a_0(t)(y - y_p)(t) = f(t) - f(t) = 0,$$

so the difference $y - y_p$ solves the homogeneous equation (8). Combined with Theorem 3.2, this observation proves the following theorem.

Theorem 3.3. *Let y_p be a solution of* (7) *(a* particular solution*), and let $\{y_1, y_2\}$ be a fundamental set of solutions of* (8)*. Then every solution of* (7) *has the form*

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t), \qquad c_1, c_2 \in \mathbb{R}.$$

Solution of the inhomogeneous ODE = particular solution + solution of the homogeneous ODE.

3.2 Constant coefficients

In this section, the coefficients are real numbers a_1 and a_0, so that the results of Section 3.1 hold with $I = \mathbb{R}$. We saw in Section 2.1 that the solutions of linear first-order equations are exponentials. (E.g, $y'(t) + 2y(t) = 0$ has general solution $y(t) = Le^{-2t}$.) Are the solutions of a second-order equation with constant coefficients also exponentials?

Example 3.6. Consider the ODE

$$y''(t) + 5y'(t) + 6y(t) = 0.$$

We make the *Ansatz* $y(t) = e^{\lambda t}$. (*Ansatz* is a German word meaning "approach to a task". In our context, it means "educated guess.") Substituting into the ODE gives

$$\left(\lambda^2 + 5\lambda + 6\right) e^{\lambda t} = 0 \quad \iff \quad (\lambda + 2)(\lambda + 3) = 0 \quad \iff \quad \lambda = -2 \text{ or } \lambda = -3.$$

We obtain two solutions, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$, with Wronskian

$$W(t) = e^{-2t}\left(-3e^{-3t}\right) - \left(-2e^{-2t}\right) e^{-3t} = -e^{-5t} \neq 0.$$

By Theorem 3.2, the general solution is $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$. Suppose we have the initial conditions $y(0) = 2$ and $y'(0) = 3$. The system (10) reads

$$\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Recall that a 2×2 matrix with nonzero determinant $ad - bc$ has the inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Here the determinant is $W(0) = -1$, and therefore

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}.$$

Hence $y(t) = 9e^{-2t} - 7e^{-3t}$.

The same computation applies to every homogeneous equation with constant coefficients,

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0. \tag{11}$$

Substituting $y(t) = e^{\lambda t}$ with $\lambda \in \mathbb{R}$ gives $(\lambda^2 + a_1\lambda + a_0)e^{\lambda t} = 0$. The exponential never vanishes, so $e^{\lambda t}$ solves (11) if and only if λ is a root of the *characteristic polynomial*

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0.$$

The roots of p are

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2},$$

and the sign of the discriminant $a_1^2 - 4a_0$ separates three cases.

3.2.1 Case 1: distinct real roots ($a_1^2 > 4a_0$)

The functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ solve (11), and their Wronskian

$$W(t) = e^{\lambda_1 t}\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} e^{\lambda_2 t} = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t}$$

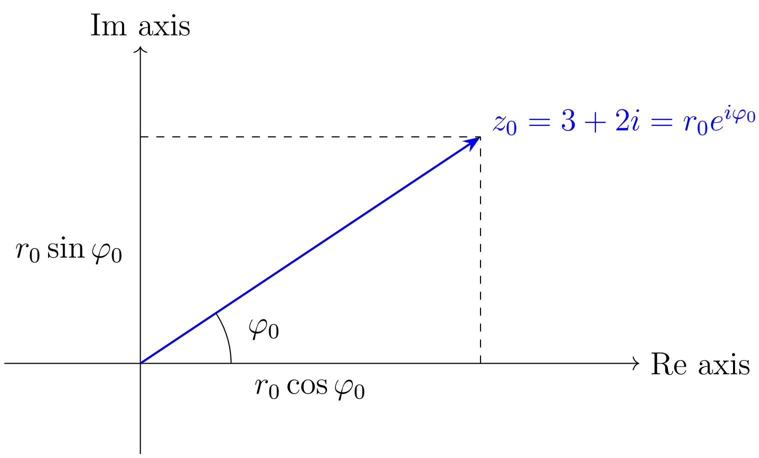
never vanishes. By Theorem 3.2, the general solution is

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.}$$

Example 3.6 falls into this case.

3.2.2 Case 2: complex roots ($a_1^2 < 4a_0$)

The roots of p are now complex numbers. We write a complex number as $z = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, where $i^2 = -1$. The number $\alpha = \operatorname{Re} z$ is the *real part* of z and the number $\beta = \operatorname{Im} z$ is the *imaginary part* of z. We picture z as the point (α, β) of the complex plane. Note we can also describe it by its distance r from the origin and its angle φ with the real axis.



For $z_0 = 3 + 2i$, we have $r_0 = \sqrt{3^2 + 2^2} = \sqrt{13}$ and $\varphi_0 = \arctan(2/3)$, so that $z_0 = r_0 \cos\varphi_0 + i r_0 \sin\varphi_0 = r_0 e^{i\varphi_0}$. The last equality rests on an important result, *Euler's formula*:

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi.}$$

For $\lambda = \alpha + i\beta$ and $t \in \mathbb{R}$, Euler's formula gives

$$e^{\lambda t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}\cos(\beta t) + i e^{\alpha t}\sin(\beta t).$$

Differentiating the real and imaginary parts separately, one checks that $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$, exactly as for real λ. Consequently, the computation above shows that the complex-valued function $e^{\lambda t}$ solves (11) whenever $p(\lambda) = 0$.

Let $\lambda = \alpha + i\beta$ with $\beta \neq 0$ be a root of p. (Because the coefficients of p are real, the other root is the complex conjugate $\bar{\lambda} = \alpha - i\beta$; here $\alpha = -a_1/2$ and $\beta = \sqrt{4a_0 - a_1^2}/2$.)

The coefficients a_1 and a_0 are real, so taking the real and imaginary parts of (11) for the complex-valued solution $e^{\lambda t}$ shows that the real and imaginary parts of $e^{\lambda t}$,

$$y_1(t) = e^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad y_2(t) = e^{\alpha t}\sin(\beta t),$$

are real solutions of (11). Their Wronskian is

$$\begin{aligned} W(t) &= e^{\alpha t}\cos(\beta t)\left(\alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t)\right) \\ &\qquad - \left(\alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t)\right) e^{\alpha t}\sin(\beta t) \\ &= \beta e^{2\alpha t}\left(\cos^2(\beta t) + \sin^2(\beta t)\right) = \beta e^{2\alpha t} \neq 0. \end{aligned}$$

By Theorem 3.2, the general solution is

$$\boxed{y(t) = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right).}$$

Example 3.7 (Harmonic oscillator). Consider the harmonic oscillator $y''(t) + \omega_0^2 y(t) = 0$ with $\omega_0 > 0$. The characteristic polynomial $p(\lambda) = \lambda^2 + \omega_0^2$ has the roots $\pm i\omega_0$, so $\alpha = 0$ and $\beta = \omega_0$, and the general solution is

$$\boxed{y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t).}$$

This confirms the general solution of the linearized pendulum stated in Example 1.1, where $\omega_0 = \sqrt{g/\ell}$, and answers the second question of Example 3.1.

3.2.3 Case 3: a double root ($a_1^2 = 4a_0$)

Now $p(\lambda) = (\lambda - \lambda_0)^2$ with $\lambda_0 = -a_1/2$, and the Ansatz produces only one solution, $e^{\lambda_0 t}$. For example, the equation $y''(t) = 0$ has $p(\lambda) = \lambda^2$ and $\lambda_0 = 0$; the Ansatz gives only the constant functions, but $y(t) = t$ is another solution. In general, for every $\lambda \in \mathbb{R}$, the function $y(t) = te^{\lambda t}$ satisfies $y'(t) = (1 + \lambda t)e^{\lambda t}$ and $y''(t) = (2\lambda + \lambda^2 t)e^{\lambda t}$, so

$$y''(t) + a_1 y'(t) + a_0 y(t) = \left[\left(\lambda^2 + a_1\lambda + a_0\right)t + (2\lambda + a_1)\right]e^{\lambda t} = \left[p(\lambda)t + p'(\lambda)\right]e^{\lambda t}.$$

At the double root, $p(\lambda_0) = 0$ and $p'(\lambda_0) = 2\lambda_0 + a_1 = 0$, so $te^{\lambda_0 t}$ solves (11). The Wronskian of $e^{\lambda_0 t}$ and $te^{\lambda_0 t}$ is

$$W(t) = e^{\lambda_0 t}(1 + \lambda_0 t)e^{\lambda_0 t} - \lambda_0 e^{\lambda_0 t} te^{\lambda_0 t} = e^{2\lambda_0 t} \neq 0,$$

and the general solution is

$$\boxed{y(t) = (c_1 + c_2 t)e^{\lambda_0 t}.}$$

Remark 3.1 (Equations of order n). The same Ansatz works for the nth-order equation with constant coefficients,

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0.$$

Substituting $y(t) = e^{\lambda t}$ leads to the characteristic polynomial $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, which has n roots in $\mathbb{C}$, counted with multiplicity, by the fundamental theorem of algebra. A root λ_j has multiplicity m if $(\lambda - \lambda_j)^m$ divides $p(\lambda)$ and $(\lambda - \lambda_j)^{m+1}$ does not. In that case, the functions $e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{m-1}e^{\lambda_j t}$ are linearly independent solutions (we leave the proof as an exercise). Taking real and imaginary parts for complex roots, as in Case 2, we obtain n linearly independent real solutions:

Root	Multiplicity	Independent solutions
$\lambda_j \in \mathbb{R}$	1	$e^{\lambda_j t}$
	$m > 1$	$e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{m-1}e^{\lambda_j t}$
$\alpha_j \pm i\beta_j$ $(\beta_j \neq 0)$	1	$e^{\alpha_j t}\cos(\beta_j t), e^{\alpha_j t}\sin(\beta_j t)$
	$m > 1$	$t^k e^{\alpha_j t}\cos(\beta_j t), t^k e^{\alpha_j t}\sin(\beta_j t),\ k = 0, \ldots, m-1$

Remark 3.2. The results of Section 3.1 extend to order n: the solution space has dimension n, the Wronskian becomes the determinant of the $n \times n$ matrix with entries $y_j^{(k)}(t)$ for $k = 0, \ldots, n-1$ and $j = 1, \ldots, n$, and Abel's identity holds with a_1 replaced by a_{n-1}. We prove these statements in the chapter on linear systems, where we rewrite the nth-order equation as a first-order system (Remark 1.1).

Example 3.8. Consider $y^{(4)}(t) + 2y''(t) - 8y'(t) + 5y(t) = 0$. Then

$$p(\lambda) = \lambda^4 + 2\lambda^2 - 8\lambda + 5 = (\lambda - 1)^2(\lambda^2 + 2\lambda + 5),$$

with four roots: $\lambda = 1$ (double), giving e^t and te^t, and $\lambda = -1 \pm 2i$, giving $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$. Hence

$$y(t) = (c_1 + c_2 t)e^t + e^{-t}\left(c_3\cos(2t) + c_4\sin(2t)\right).$$

3.3 Solving inhomogeneous equations

We return to the inhomogeneous equation (7). By Theorem 3.3, it suffices to find one particular solution y_p. How do we find particular solutions? We present two methods: the method of undetermined coefficients in Section 3.3.1 and the method of variation of constants in Section 3.3.2. The Laplace transform, in the next chapter, gives a third.

3.3.1 Undetermined coefficients

Consider the inhomogeneous equation with constant coefficients,

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t). \tag{12}$$

The method of undetermined coefficients applies when f is a polynomial, an exponential e^{rt}, a sine or cosine, or a product of these. We guess a particular solution of the same form as f with unknown coefficients, substitute it into (12), and solve for the coefficients. For example, if f is a polynomial of degree d, we try a polynomial of degree d; if $f(t) = e^{rt}$, we try Ce^{rt}; and if $f(t) = \cos(\omega t)$ or $f(t) = \sin(\omega t)$, we try $A\cos(\omega t) + B\sin(\omega t)$. If the trial form contains a solution of the homogeneous equation, it cannot produce f. In that case, we multiply the trial form by t^m, where m is the multiplicity of the number 0, r, or $i\omega$, respectively, as a root of the characteristic polynomial p.

Example 3.9. Consider the ODE $y''(t) - 2y(t) = 5t$. The inhomogeneity is a polynomial of degree 1, and 0 is not a root of $p(\lambda) = \lambda^2 - 2$, so we try $y_p(t) = A_0 + A_1 t$. Then $y_p''(t) = 0$, and substitution gives

$$-2A_0 - 2A_1 t = 5t \quad \text{for every } t \in \mathbb{R} \quad \Longleftrightarrow \quad A_0 = 0 \text{ and } A_1 = -\frac{5}{2}.$$

Hence $y_p(t) = -\frac{5}{2}t$. The roots of p are $\sqrt{2}$ and $-\sqrt{2}$, so the homogeneous solutions are $e^{\sqrt{2}t}$ and $e^{-\sqrt{2}t}$, and by Theorem 3.3,

$$y(t) = -\frac{5t}{2} + c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

Section 3.5 works out a case in which the trial form must be multiplied by t: the oscillator forced at its natural frequency.

3.3.2 Variation of constants

Variation of constants is a general technique for inhomogeneous linear ODEs, with constant or nonconstant coefficients; it is due to Joseph-Louis Lagrange (ca. 1776). We return to the setting of Section 3.1: the functions a_1, a_0, and f are continuous on I, and $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation (8). As in Section 2.2, where the constant L became a function $L(t)$, we look for a solution of (7) of the form

$$y(t) = c_1(t)y_1(t) + c_2(t)y_2(t),$$

with differentiable functions c_1 and c_2 in place of constants. Then

$$y'(t) = c_1'(t)y_1(t) + c_2'(t)y_2(t) + c_1(t)y_1'(t) + c_2(t)y_2'(t).$$

We impose the condition $c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$, so that $y'(t) = c_1(t)y_1'(t) + c_2(t)y_2'(t)$ and

$$y''(t) = c_1'(t)y_1'(t) + c_2'(t)y_2'(t) + c_1(t)y_1''(t) + c_2(t)y_2''(t).$$

Substituting into the left-hand side of (7) gives

$$y'' + a_1 y' + a_0 y = c_1' y_1' + c_2' y_2' + c_1\left(y_1'' + a_1 y_1' + a_0 y_1\right) + c_2\left(y_2'' + a_1 y_2' + a_0 y_2\right) = c_1' y_1' + c_2' y_2',$$

because y_1 and y_2 solve (8). Therefore, y solves (7) whenever c_1' and c_2' satisfy

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad t \in I.$$

The determinant of this matrix is the Wronskian $W(t)$, which does not vanish on I by Proposition 3.2. Cramer's rule gives

$$c_1'(t) = -\frac{y_2(t)f(t)}{W(t)}, \qquad c_2'(t) = \frac{y_1(t)f(t)}{W(t)}.$$

Integrating from $t_0 \in I$ to t and inserting the result into $y = c_1 y_1 + c_2 y_2$ yields the particular solution

$$\boxed{y_p(t) = \int_{t_0}^{t} \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{W(\tau)} f(\tau)\, d\tau.}$$

Adding constants of integration $\tilde{c}_1$ and $\tilde{c}_2$ to c_1 and c_2 adds $\tilde{c}_1 y_1 + \tilde{c}_2 y_2$ to y_p and recovers the general solution of Theorem 3.3.

Procedure:

(1) Solve the linear system for $c_1'(t)$ and $c_2'(t)$.

(2) Integrate to obtain $c_1(t)$ and $c_2(t)$.

(3) Then $y(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$.

Example 3.10. Consider the ODE $y''(t) - 5y'(t) + 6y(t) = 2e^t$.

(1) (Fundamental set of solutions.) The characteristic polynomial is $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$, which gives $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$, with Wronskian

$$W(t) = e^{3t} \cdot 2e^{2t} - 3e^{3t} \cdot e^{2t} = -e^{5t}.$$

(2) (Variation of constants.) Cramer's rule gives

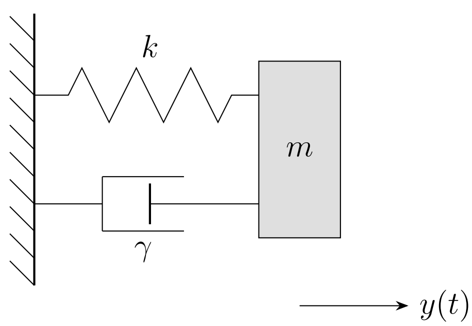
$$c_1'(t) = -\frac{e^{2t} \cdot 2e^t}{-e^{5t}} = 2e^{-2t}, \qquad c_2'(t) = \frac{e^{3t} \cdot 2e^t}{-e^{5t}} = -2e^{-t},$$

so $c_1(t) = -e^{-2t} + \tilde{c}_1$ and $c_2(t) = 2e^{-t} + \tilde{c}_2$ with constants $\tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$. Therefore,

$$y(t) = \left(\tilde{c}_1 - e^{-2t}\right) e^{3t} + \left(\tilde{c}_2 + 2e^{-t}\right) e^{2t} = \tilde{c}_1 e^{3t} + \tilde{c}_2 e^{2t} + e^t.$$

3.4 Free vibrations: the damped oscillator

Consider a mass $m > 0$ attached to a spring with spring constant $k > 0$, whose other end is fixed to a wall, and let $y(t)$ denote the displacement of the mass from its equilibrium position. The spring exerts the restoring force $-ky(t)$ (Hooke's law). A damper (a dashpot or the resistance of the surrounding air) exerts a force $-\gamma y'(t)$, proportional to the velocity, with damping coefficient $\gamma \geq 0$.



By Newton's second law, $my''(t) = -ky(t) - \gamma y'(t)$, that is,

$$my''(t) + \gamma y'(t) + ky(t) = 0. \tag{13}$$

When $\gamma = 0$, equation (13) is the harmonic oscillator of Example 3.7 with natural frequency $\omega_0 = \sqrt{k/m}$. When $\gamma > 0$, the characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$ has the roots

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m},$$

and the three cases of Section 3.2 correspond to three physical regimes.

(1) *Overdamped* ($\gamma^2 > 4km$). The roots are real, distinct, and negative, since $\sqrt{\gamma^2 - 4km} < \gamma$. The general solution is $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. A nonzero solution vanishes at most once, because $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = 0$ is equivalent to $e^{(\lambda_1 - \lambda_2)t} = -c_2/c_1$ when $c_1 \neq 0$; the mass does not oscillate.

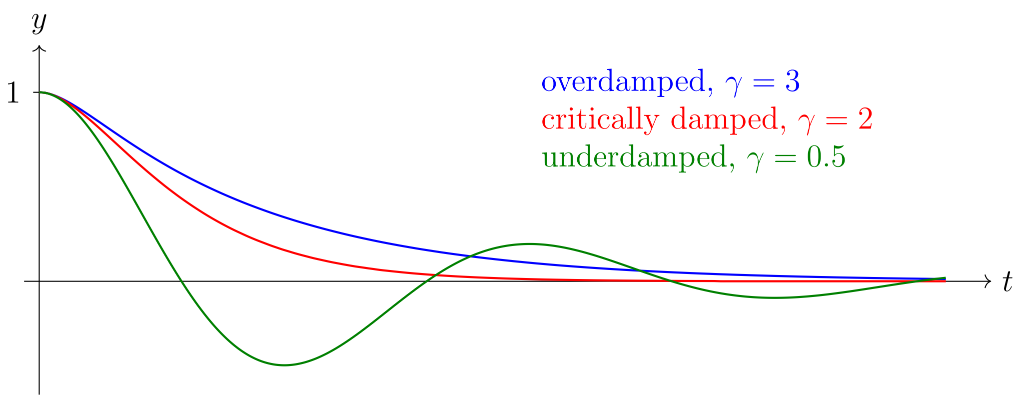
(2) *Critically damped* ($\gamma^2 = 4km$). The double root is $\lambda_0 = -\gamma/(2m)$, and the general solution is $y(t) = (c_1 + c_2 t)e^{-\gamma t/(2m)}$.

(3) *Underdamped* ($\gamma^2 < 4km$). The roots are $-\gamma/(2m) \pm i\omega_d$, where

$$\omega_d = \frac{\sqrt{4km - \gamma^2}}{2m} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}},$$

and the general solution is $y(t) = e^{-\gamma t/(2m)}\left(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\right)$. The mass oscillates with frequency $\omega_d < \omega_0$ and an amplitude that decays exponentially.

In all three regimes, every solution tends to 0 as $t \to \infty$, because each root has negative real part (and $te^{-\gamma t/(2m)} \to 0$ in the critically damped case). The figure below shows the solution with $m = k = 1$, $y(0) = 1$, and $y'(0) = 0$ for $\gamma = 3$ (overdamped), $\gamma = 2$ (critically damped), and $\gamma = 0.5$ (underdamped).



Example 3.11 (Electrical circuits, revisited). The series circuit of Example 1.2 with $V \equiv 0$, namely $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0$, has the same form as (13): the inductance L plays the role of the mass m, the resistance R that of the damping coefficient γ, the inverse capacitance $1/C$ that of the spring constant k, and the charge Q that of the displacement y. Everything we showed about (13) therefore applies to the circuit. For example, the circuit is underdamped, and the charge oscillates, if and only if $R^2 < 4L/C$. (The source V corresponds to an external force; see Section 3.5.)

3.5 Forced vibrations and resonance

Suppose now that an external force $F(t)$ acts on the mass of Section 3.4. Newton's second law gives $my''(t) = F(t) - ky(t) - \gamma y'(t)$, and dividing by m,

$$y''(t) + \frac{\gamma}{m}y'(t) + \omega_0^2 y(t) = f(t), \qquad f(t) = \frac{F(t)}{m}.$$

We consider a periodic force $F(t) = F_0 \cos(\omega t)$ with amplitude $F_0 > 0$ and forcing frequency $\omega > 0$.

3.5.1 Undamped forcing and beats

Let $\gamma = 0$, so that

$$y''(t) + \omega_0^2 y(t) = \frac{F_0}{m}\cos(\omega t). \tag{14}$$

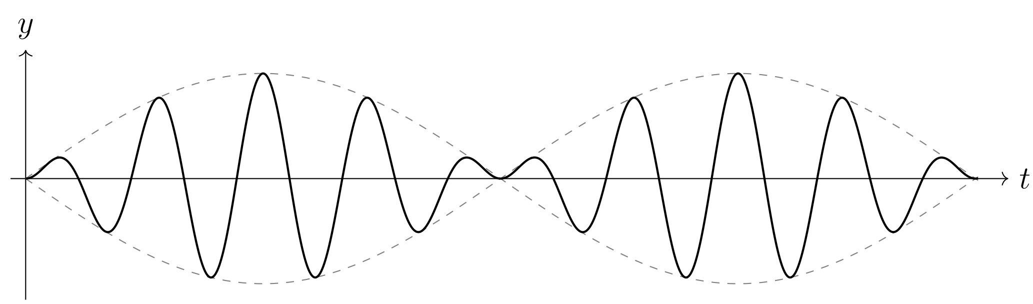
Suppose first that $\omega \neq \omega_0$. Then $i\omega$ is not a root of $p(\lambda) = \lambda^2 + \omega_0^2$, and we try $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Substitution gives $(\omega_0^2 - \omega^2)(A\cos(\omega t) + B\sin(\omega t)) = \frac{F_0}{m}\cos(\omega t)$, so

$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t).$$

Suppose that the mass starts at rest, $y(0) = y'(0) = 0$. The general solution is $y(t) = y_p(t) + c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$, and the initial conditions give $c_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}$ and $c_2 = 0$. The identity $\cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2}$ then yields

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{(\omega_0 - \omega)t}{2}\right)\sin\left(\frac{(\omega_0 + \omega)t}{2}\right).$$

When ω is close to ω_0, the second sine oscillates fast, while the first sine varies slowly and modulates the amplitude between 0 and $\frac{2F_0}{m|\omega_0^2-\omega^2|}$. This phenomenon is called *beats*. The figure below shows the solution for $\omega_0 = 1$, $\omega = 0.8$, and $F_0/m = 1$, together with the envelope $\pm\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{(\omega_0-\omega)t}{2}\right)$ (dashed).



3.5.2 Resonance

Suppose now that $\omega = \omega_0$, so that the forcing frequency equals the natural frequency:

$$y''(t) + \omega_0^2 y(t) = \frac{F_0}{m}\cos(\omega_0 t).$$

Now $i\omega_0$ is a simple root of p, so the rule of Section 3.3.1 tells us to multiply the trial form by t: let us try $y_p(t) = t\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$. Here,

$$\begin{aligned} y_p'(t) &= A\cos(\omega_0 t) + B\sin(\omega_0 t) + t\left(-A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t)\right), \\ y_p''(t) &= 2\left(-A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t)\right) - t\left(A\omega_0^2\cos(\omega_0 t) + B\omega_0^2\sin(\omega_0 t)\right), \end{aligned}$$

so

$$y_p''(t) + \omega_0^2 y_p(t) = 2\left(-A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t)\right).$$

This expression equals $\frac{F_0}{m}\cos(\omega_0 t)$ for every t if and only if $A = 0$ and $B = \frac{F_0}{2m\omega_0}$. Therefore, $y_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$, and the general solution is

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

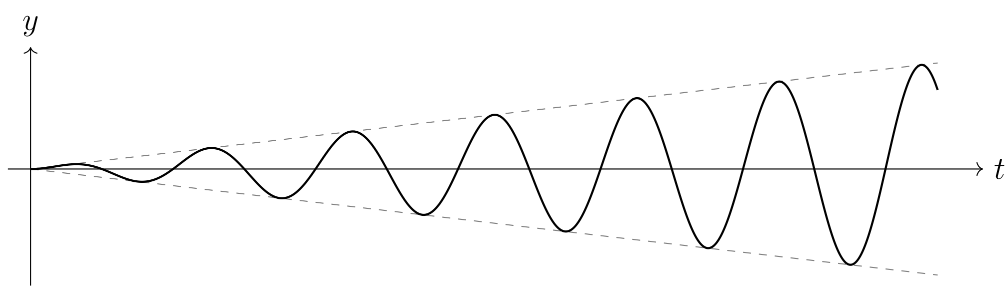
Suppose again that the mass starts at rest, $y(0) = y'(0) = 0$. Then $y(0) = c_1 = 0$ and $y'(0) = \omega_0 c_2 = 0$, so the unique solution is

$$y(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

Note: $\limsup_{t\to\infty} |y(t)| = \infty$ (!). This behavior corresponds to *harmonic resonance*, a phenomenon that occurs when the frequency of a periodic forcing is equal, or nearly equal, to a natural frequency of the system. (E.g., a swing at a playground, acoustic resonances of musical instruments...) The resonant solution is also the limit of the beats: for fixed t, l'Hôpital's rule in the variable ω gives

$$\lim_{\omega\to\omega_0} \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2} = \lim_{\omega\to\omega_0} \frac{-t\sin(\omega t)}{-2\omega} = \frac{t\sin(\omega_0 t)}{2\omega_0}.$$

The figure below shows the resonant solution for $\omega_0 = 1$ and $F_0/m = 1$, together with the envelope $\pm\frac{F_0}{2m\omega_0}t$ (dashed).



3.5.3 Damped forcing

Let $\gamma > 0$, so that

$$y''(t) + \frac{\gamma}{m}y'(t) + \omega_0^2 y(t) = \frac{F_0}{m}\cos(\omega t).$$

The roots of the characteristic polynomial have negative real part (Section 3.4), so $i\omega$ is not a root, and we try $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Comparing the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ after substitution gives

$$(\omega_0^2 - \omega^2)A + \frac{\gamma\omega}{m}B = \frac{F_0}{m}, \qquad -\frac{\gamma\omega}{m}A + (\omega_0^2 - \omega^2)B = 0,$$

whose solution is

$$A = \frac{F_0}{m}\frac{\omega_0^2 - \omega^2}{\Delta}, \qquad B = \frac{F_0}{m}\frac{\gamma\omega/m}{\Delta}, \qquad \text{where } \Delta = (\omega_0^2 - \omega^2)^2 + \left(\frac{\gamma\omega}{m}\right)^2 > 0.$$

We can write this particular solution as $y_p(t) = R\cos(\omega t - \delta)$, with amplitude $R = \sqrt{A^2 + B^2}$ and phase δ determined by $\cos\delta = A/R$ and $\sin\delta = B/R$:

$$R = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega/m)^2}}.$$

By Theorem 3.3, every solution has the form $y = y_p + y_h$, where y_h solves the damped equation (13) and therefore tends to 0 as $t \to \infty$. We call y_h the *transient* and y_p the *steady state*: whatever the initial conditions, every solution approaches the steady state, which oscillates with the forcing frequency ω. Unlike the undamped case, the amplitude R stays bounded for every $\omega > 0$. Minimizing the denominator of R over ω^2 shows that, if $\omega_0^2 > \gamma^2/(2m^2)$, the amplitude is largest at

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}}, \qquad \text{with } R(\omega_{\max}) = \frac{F_0}{\gamma\omega_d};$$

otherwise, R decreases as ω increases. As $\gamma \to 0$, the frequency $\omega_{\max}$ tends to ω_0 and the peak amplitude $F_0/(\gamma\omega_d)$ grows without bound, in line with the resonance of the undamped oscillator.

Summary.

1) The solutions of a second-order linear homogeneous ODE form a vector space of dimension two. Two solutions form a fundamental set if and only if their Wronskian is nonzero at one point, and Abel's identity shows that the Wronskian then vanishes nowhere.

2) Every solution of the inhomogeneous ODE is a particular solution plus a solution of the homogeneous ODE.

3) For constant coefficients, the roots of the characteristic polynomial give a fundamental set: $e^{\lambda_1 t}, e^{\lambda_2 t}$ for distinct real roots, $e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)$ for complex roots $\alpha \pm i\beta$, and $e^{\lambda_0 t}, te^{\lambda_0 t}$ for a double root.

4) Undetermined coefficients produce particular solutions for polynomial, exponential, and trigonometric inhomogeneities; variation of constants produces them for every continuous inhomogeneity.

5) The damped oscillator is overdamped, critically damped, or underdamped according to the sign of $\gamma^2 - 4km$; when $\gamma > 0$, every solution tends to 0.

6) Periodic forcing of the undamped oscillator produces beats when $\omega \neq \omega_0$ and unbounded resonant growth when $\omega = \omega_0$; with damping, every solution approaches a bounded steady state.