MATH 441: Differential Equations

Lecture notes prepared by Gabriel P. Langlois

Fall 2026

Contents

1 Introduction

What does the name "ordinary differential equation" (ODE) mean?

- **O**rdinary: in one independent variable.
- **D**ifferential: derivatives of an unknown function appear.
- **E**quation: the unknown function and its derivatives are related by an equation.

1.1 What is an ODE?

Definition 1.1. An *ordinary differential equation* (ODE) is an equation for an unknown function $y\colon I \subseteq \mathbb{R} \to \mathbb{R}$ that relates y, its derivatives, and the independent variable:

$$\Phi\left(\frac{d^n y}{dt^n}(t), \frac{d^{n-1}y}{dt^{n-1}}(t), \ldots, \frac{dy}{dt}(t), y(t), t\right) = 0 \tag{1}$$

via some function Φ.

- The number $n \geq 1$ is called the *order* of the ODE.
- The function y is called the *dependent variable* (it depends on t).
- The variable t is called the *independent variable*.

There is no single uniform notation for differentiation.

- Leibniz's notation: $\frac{dy}{dt}(t)$;
- Lagrange's notation: $y'(t)$;
- Newton's notation: $\dot{y}(t)$ (used mainly when the independent variable denotes time).

1.2 ODEs as models of scientific phenomena

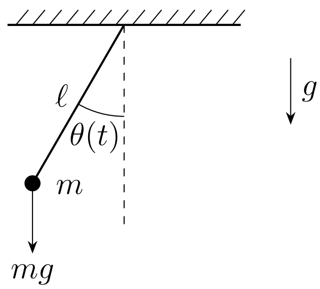
The workflow of mathematical modeling is: application $\to$ model $\to$ ODE $\to$ solutions. We use ODEs to model processes in

- physics and engineering (mass-spring systems, aircraft, climate prediction);
- chemistry (reaction networks and chemical dynamics);
- biology (epidemic models, cancer and tumor growth, ecology);
- economics.

Here are three important examples. We will revisit them later in the course.

Example 1.1 (The pendulum). We model a pendulum with the following quantities:

- t: time;
- $\theta(t)$: angle at time t;
- m: mass of the pendulum;
- ℓ: length of the pendulum;
- g: gravity constant.



The model comes from Newton's second law of motion, $F(t) = ma(t)$, where the tangential acceleration of the bob is $a(t) = \ell\frac{d^2\theta}{dt^2}(t)$ and the tangential component of gravity provides the restoring force $F(t) = -mg\sin(\theta(t))$. The force balance along the tangential direction gives

$$m\ell\frac{d^2\theta}{dt^2}(t) = -mg\sin(\theta(t)),$$

and therefore

$$\boxed{\frac{d^2\theta}{dt^2}(t) + \omega_0^2\sin(\theta(t)) = 0,}$$

where $\omega_0^2 = g/\ell$ and the number $\omega_0 = \sqrt{g/\ell}$ is the angular frequency. This is a nonlinear equation, and it cannot be solved in terms of elementary functions! However, we have that $\sin(\theta(t)) \approx \theta(t)$ for small angles (recall the Maclaurin expansion $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$), and this leads to the linearized ODE (the *harmonic oscillator*)

$$\boxed{\frac{d^2\theta}{dt^2}(t) + \omega_0^2\theta(t) = 0.}$$

The general solution is $\theta(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, where C_1 and C_2 are (a priori undetermined) numbers.

Example 1.2 (Electrical circuits). Consider a series circuit with an inductor of inductance L, a resistor of resistance R, a capacitor of capacitance C, and a voltage source $V(t)$. Let $Q(t)$ denote the charge on the capacitor, so that the current is $I(t) = Q'(t)$. Kirchhoff's voltage law states that the voltage drops across the three elements balance the source:

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = V(t).$$

This is a second-order linear ODE with inhomogeneous term $t \mapsto V(t)$.

Example 1.3 (Population growth). Let $N(t)$ denote the size of a population at time t and let $r > 0$ denote its growth rate. The simplest model (Malthus, 1798) posits that the population grows in proportion to its size:

$$\dot{N}(t) = rN(t), \qquad N(0) = N_0 > 0.$$

Its solution is $N(t) = N_0 e^{rt}$, so the model predicts unbounded exponential growth (unrealistic...). A better model (Verhulst, 1838) caps the growth at a *carrying capacity* K:

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

What do we mean by "solving" an ODE? We distinguish between

- analytical solutions,
- solution properties (qualitative behavior),
- numerical solutions.

In this course, we focus on the analytical aspects of ODEs. We also discuss applications of ODEs, study qualitative behavior, and introduce numerical solutions in the final chapter.

1.3 Classification of ODEs

Order. The order of an ODE is the order n of the highest derivative that appears in (1).

Linear versus nonlinear. An ODE is called *linear* if it has the form

$$a_n(t)\frac{d^n y}{dt^n}(t) + \cdots + a_1(t)\frac{dy}{dt}(t) + a_0(t)y(t) = f(t),$$

with $a_n \not\equiv 0$. Otherwise, the ODE is nonlinear. For a linear ODE, we further say that the equation is *homogeneous* if $f \equiv 0$ and *inhomogeneous* if $f \not\equiv 0$. Among the examples above, the harmonic oscillator and the circuit equation are linear, while the pendulum equation and the logistic equation are nonlinear.

Autonomous versus nonautonomous. An ODE is called *autonomous* if the independent variable does not appear explicitly, as in the pendulum and logistic equations.

Scalar equations versus systems. An ODE for a single unknown function is a *scalar* equation. Several coupled ODEs for several unknown functions form a *system.*

Example 1.4 (Lotka–Volterra predator–prey model). Let $x(t)$ denote a prey population and $y(t)$ a predator population. To describe how they interact, Lotka (1925) and Volterra (1926) proposed the coupled nonlinear system

$$\frac{dx}{dt}(t) = \alpha x(t) - \beta x(t)y(t), \qquad \frac{dy}{dt}(t) = \delta x(t)y(t) - \gamma y(t).$$

Here α is the prey's growth rate and γ the predators' death rate, so in isolation the prey grows and the predators decay exponentially. The bilinear terms model encounters between the species: predation removes prey at rate β and sustains predators at rate δ.

Remark 1.1. An nth-order ODE in explicit form, $y^{(n)}(t) = F(t, y(t), \dots, y^{(n-1)}(t))$, can be transformed into a system of n first-order ODEs.

Example 1.5. Consider the harmonic oscillator $\frac{d^2\theta}{dt^2}(t) + \omega_0^2\theta(t) = 0$. Let $y_1(t) = \theta(t)$ and $y_2(t) = \frac{d\theta}{dt}(t)$. Then

$$\frac{dy_1}{dt}(t) = y_2(t), \qquad \frac{dy_2}{dt}(t) = -\omega_0^2 y_1(t),$$

which can be written in matrix form as

$$\begin{pmatrix} \frac{dy_1}{dt}(t) \\ \frac{dy_2}{dt}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

Initial value problems versus boundary value problems. Consider the ODE

$$\frac{dy}{dt}(t) = 2t \quad \Longrightarrow \quad y(t) = t^2 + C \quad \text{for any number } C.$$

This ODE has infinitely many solutions! We need additional conditions to make the solution unique. For example, the conditions $\frac{dy}{dt}(t) = 2t$ and $y(0) = 4$ yield $y(t) = t^2 + 4$. Typically, an nth-order ODE requires n additional conditions. We can specify them in different ways.

a) For an nth-order ODE, prescribe the solution and its first $n-1$ derivatives at one point (an evolution process). For example, the ODE

$$\frac{d^3y}{dt^3}(t) + t\frac{d^2y}{dt^2}(t) + 9\frac{dy}{dt}(t) + 9ty(t) = 0 \quad \text{with} \quad y(0) = 2, \; \frac{dy}{dt}(0) = 0, \; \frac{d^2y}{dt^2}(0) = -18$$

has the unique solution $y(t) = 2\cos(3t)$. This is called an *initial value problem* (IVP).

b) Prescribe the solution at different points (for a second-order ODE, at two different points). For example, the ODE

$$\frac{d^2y}{dx^2}(x) + y(x) = \cos(x) \quad \text{with} \quad y(0) = y(1) = 0$$

has the unique solution $y(x) = \frac{1}{2}(x-1)\sin x$. This is called a *boundary value problem*.

Remark 1.2 (Beyond ODEs). There are other types of differential equations. For example:

a) Partial differential equations (PDEs): differential equations for functions of more than one independent variable. For example, the *heat equation*: let $u(x,t)$ denote the temperature of a rod at position x and time t. The 1D heat PDE is

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0,$$

where $u(x,0)$ is the initial temperature.

b) Integro-differential equations (IDEs). For example, a linear *Volterra integro-differential equation*:

$$y'(t) + \int_0^t K(t,s)y(s)\,ds = f(t),$$

where K and f are given. (This is used, e.g., in polymer physics, ecology, water waves, and quantum mechanics.)

c) Delay differential equations (DDEs): differential equations in which the derivative at time t depends on the value of the function at an earlier time. For example, the *Mackey–Glass equation*:

$$\frac{dx}{dt}(t) = \frac{\beta x(t-\tau)}{1 + x(t-\tau)^n} - \gamma x(t),$$

where β, γ, and τ are real numbers and n is a positive integer. (Michael C. Mackey and Leon Glass introduced this equation in 1977 to model complex dynamics in mammalian physiology.)

Summary.

1) An ODE is an equation for an unknown function *in one variable* that involves the derivative(s) of that function; its order is the order of the highest derivative.

2) ODEs arise as models throughout the sciences; the pendulum, population growth, and electrical circuits are our recurring examples.

3) We classify ODEs by order, linearity (and homogeneity), autonomy, scalar versus system, and by the type of side conditions (initial value problem vs. boundary value problem).

2 First-order ODEs

We begin with the general first-order ODE

$$y'(t) = f(t, y(t)),$$

where f is a given function of two variables. This chapter develops the elementary solution methods (separation of variables, variation of constants, exact equations), the qualitative theory on the line (the phase line), and the fundamental existence and uniqueness theorem, whose proof we defer to the chapter on nonlinear systems. We close the chapter with a first look at numerical approximation, namely *Euler's method*.

2.1 Separable equations

Example 2.1. Consider the initial value problem

$$y'(t) = -2ty(t)^2, \qquad y(0) = 1.$$

Suppose, for now, that $y(t) \neq 0$ for every $t > 0$. Divide the equation by $y(t)^2$ and write

$$\frac{y'(t)}{y(t)^2} = -2t \quad \Longleftrightarrow \quad \frac{d}{dt}\left(-\frac{1}{y(t)}\right) = \frac{d}{dt}\left(-t^2\right).$$

Integrating both sides with respect to the independent variable t yields $-1/y(t) = -t^2 + C$. The initial condition $y(0) = 1$ forces $C = -1$, and therefore

$$y(t) = \frac{1}{1+t^2}.$$

This function is positive and differentiable on all of $\mathbb{R}$, which justifies the division, and we can substitute into the equation to confirm it solves the ODE.

Note in this example that the right-hand side splits into a factor that depends on t alone and a factor that depends on y alone. We call such ODEs *separable*.

Definition 2.1. A first-order ODE is *separable* if it takes the form

$$y'(t) = f(t)g(y(t)), \tag{2}$$

where f and g are given functions of one real variable.

Let f and g be continuous, let y solve (2) on an interval I containing t_0, and suppose $g(y(t)) \neq 0$ for every $t \in I$.[1] Dividing (2) by $g(y(t))$ gives $\frac{1}{g(y(t))}y'(t) = f(t)$. We can invoke the fundamental theorem of calculus to express each side as a derivative with respect to t:

$$\frac{1}{g(y(t))}y'(t) = \frac{d}{dt}\int_{y(t_0)}^{y(t)} \frac{1}{g(z)}\,dz \qquad \text{and} \qquad f(t) = \frac{d}{dt}\int_{t_0}^{t} f(\tau)\,d\tau.$$

[1] If $g(y_*) = 0$, then the constant function $y \equiv y_*$ satisfies $y'(t) = 0 = f(t)g(y_*)$ and solves (2).

The two integrals have the same derivative on I and both vanish at $t = t_0$, so they agree:

$$\int_{y(t_0)}^{y(t)} \frac{1}{g(z)}\, dz = \int_{t_0}^{t} f(\tau)\, d\tau. \tag{3}$$

Equation (3) determines y implicitly, and the free parameter is the initial value $y(t_0)$.

Example 2.2. Consider the separable ODE

$$y'(t) = \left(1 + y(t)^2\right)\cos t, \qquad y(0) = 0,$$

which is separable with $f(t) = \cos t$ and $g(z) = 1 + z^2$. The function g never vanishes, so (3) applies and gives

$$\int_0^{y(t)} \frac{1}{1+z^2}\, dz = \int_0^t \cos\tau\, d\tau \quad \Longleftrightarrow \quad \arctan(y(t)) = \sin t.$$

Here arctan maps $\mathbb{R}$ onto $(-\pi/2, \pi/2)$, and $|\sin t| \le 1 < \pi/2$ for every t. The right-hand side therefore stays in the range of arctan, so the solution is well-defined and we find

$$y(t) = \tan(\sin t).$$

Example 2.3 (Leibniz equation). Let $a > 0$ be a constant and consider the following separable ODE due to Leibniz:

$$y'(t) = \underbrace{1}_{f(t)} \cdot \underbrace{-\frac{y(t)}{\sqrt{a^2 - y(t)^2}}}_{g(y(t))}.$$

Separation of variables gives

$$-\int_{y(t_0)}^{y(t)} \frac{\sqrt{a^2 - z^2}}{z}\, dz = t - t_0.$$

The integral can be solved using various techniques. **Hint:** Use the substitution $v^2 = a^2 - z^2$ in the integral.

An important special case is the linear homogeneous equation $y'(t) = a(t)y(t)$, for which $g(z) = z$. Separating variables gives

$$\int_{y(t_0)}^{y(t)} \frac{1}{z}\, dz = \int_{t_0}^{t} a(\tau)\, d\tau,$$

and the left-hand side equals $\ln(y(t)/y_0)$, where $y_0 := y(t_0)$, provided $y > 0$. Hence

$$y(t) = y_0 e^{\int_{t_0}^t a(\tau)\, d\tau}.$$

In other words, every such solution has the form

$$\boxed{y(t) = Le^{\int_{t_0}^t a(\tau)\,d\tau}}$$

with the constant $L = y(t_0)$ determined by the initial condition.

The computation assumed $y > 0$, but the conclusion holds for every solution. Indeed, let y solve $y'(t) = a(t)y(t)$ and set $u(t) = y(t)e^{-\int_{t_0}^t a(\tau)\,d\tau}$. Then

$$u'(t) = \left(y'(t) - a(t)y(t)\right)e^{-\int_{t_0}^t a(\tau)\,d\tau} = 0,$$

so u is constant and $y(t) = y(t_0)e^{\int_{t_0}^t a(\tau)\,d\tau}$.

2.2 Linear equations and variation of constants

Consider the linear inhomogeneous ODE

$$y'(t) = a(t)y(t) + g(t).$$

We can solve this ODE using a technique called *variation of constants* due to Johann Bernoulli (1697). The idea is to replace the constant L in the homogeneous solution with a function $t \mapsto L(t)$. First, we write

$$y(t) = L(t)e^{\int_{t_0}^t a(\tau)\,d\tau} \qquad \text{with} \qquad R(t) = e^{\int_{t_0}^t a(\tau)\,d\tau}.$$

Then $y'(t) = L'(t)R(t) + L(t)R'(t)$, with

$$R'(t) = \frac{d}{dt}\left(e^{\int_{t_0}^t a(\tau)\,d\tau}\right) = a(t)e^{\int_{t_0}^t a(\tau)\,d\tau} = a(t)R(t).$$

Hence $L(t)R'(t) = a(t)\left(L(t)R(t)\right) = a(t)y(t)$, and therefore $y'(t) = a(t)y(t) + L'(t)R(t)$. Comparing this identity with the ODE $y'(t) = a(t)y(t) + g(t)$ gives

$$L'(t)R(t) = g(t) \quad \Longrightarrow \quad L(t) = \int_{t_0}^t \frac{g(\tau)}{R(\tau)}\,d\tau + C,$$

where the division by R is okay because $R(t) = e^{\int_{t_0}^t a(\tau)\,d\tau} > 0$. It follows that

$$\boxed{y(t) = \left(\int_{t_0}^t e^{-\int_{t_0}^\tau a(\tau')\,d\tau'} g(\tau)\,d\tau + C\right)e^{\int_{t_0}^t a(\tau)\,d\tau}.}$$

The factor $e^{-\int_{t_0}^t a(\tau)\,d\tau}$ is called the *integrating factor* of the ODE $y'(t) - a(t)y(t) = g(t)$.

Example 2.4. Consider the ODE $y'(t) - \sin(t)y(t) = 1$. In the notation above, the coefficient is $a(t) = \sin t$ and the inhomogeneity is $g(t) = 1$, both continuous on $\mathbb{R}$. The coefficient has an elementary antiderivative,

$$\int_{t_0}^t a(\tau)\,d\tau = \int_{t_0}^t \sin(\tau)\,d\tau = \cos(t_0) - \cos(t),$$

so we can write the homogeneous factor $e^{\int_{t_0}^t a(\tau)\,d\tau}$ in closed form. Rather than quote the boxed formula, we run the method in three steps.

Step 1: Solve the homogeneous ODE $y'(t) - \sin(t)y(t) = 0$. We get $\frac{1}{y(t)}y'(t) = \sin(t)$, so

$$\int_{y(t_0)}^{y(t)} \frac{1}{z}\,dz = \int_{t_0}^{t} \sin(\tau)\,d\tau \quad\Longrightarrow\quad \ln(y(t)) = -\cos(t) + C \quad\Longrightarrow\quad y(t) = Le^{-\cos(t)}.$$

Step 2: Let $y(t) = L(t)e^{-\cos t}$ *be the solution to the inhomogeneous ODE.* Then

$$y'(t) = L'(t)e^{-\cos t} + L(t)e^{-\cos t}\sin(t) = L'(t)e^{-\cos t} + \sin(t)y(t),$$

and thus $y'(t) - \sin(t)y(t) = L'(t)e^{-\cos t}$.

Step 3: Plug into the ODE. We get

$$L'(t)e^{-\cos t} = 1 \quad\Longrightarrow\quad L(t) = \int_{t_0}^{t} e^{\cos\tau}\,d\tau + C \quad\Longrightarrow\quad y(t) = \left(\int_{t_0}^{t} e^{\cos\tau}\,d\tau + C\right)e^{-\cos t}.$$

The remaining integral has no elementary antiderivative, so the solution stays in this form.

2.3 Exact equations

Consider an ODE of the form

$$M(t, y(t)) + N(t, y(t))y'(t) = 0,$$

where M and N are two continuously differentiable (C^1) functions of two variables on an open rectangle $\Omega \subseteq \mathbb{R}^2$. Assume that $\frac{\partial M}{\partial y}(t, y) = \frac{\partial N}{\partial t}(t, y)$ on Ω. We call such ODEs *exact* or *total*. To solve an exact ODE, we look for a *potential function* $V\colon \Omega \to \mathbb{R}$ such that $\frac{\partial V}{\partial t} = M$ and $\frac{\partial V}{\partial y} = N$ on Ω. On a rectangle, such a V always exists (e.g., Example (2.5) below). In this case, we can write

$$\begin{aligned} \frac{d}{dt}\left(V(t, y(t))\right) &= \frac{\partial V}{\partial t}(t, y(t)) + \frac{\partial V}{\partial y}(t, y(t))y'(t) && \text{(Multivariable chain rule)} \\ &= M(t, y(t)) + N(t, y(t))y'(t) && \text{(By assumption)} \\ &= 0. \end{aligned}$$

It follows that $V(t, y(t)) = \text{constant}$. We can then solve explicitly or implicitly for $y(t)$.

Example 2.5. Consider the ODE

$$(2ty + 1) + (t^2 + 2y)y'(t) = 0.$$

Here, $M(t, y) = 2ty + 1$ and $N(t, y) = t^2 + 2y$, and $\frac{\partial M}{\partial y} = 2t = \frac{\partial N}{\partial t}$, so the equation is exact on $\Omega = \mathbb{R}^2$. Integrating $\frac{\partial V}{\partial t} = 2ty + 1$ in t gives $V(t, y) = t^2y + t + \varphi(y)$; then $\frac{\partial V}{\partial y} = t^2 + \varphi'(y) = t^2 + 2y$ forces $\varphi(y) = y^2$. Hence the solutions satisfy

$$t^2y(t) + t + y(t)^2 = C.$$

What happens when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$? We look for a nonvanishing function $\mu\colon \Omega \to \mathbb{R}$ such that

$$\frac{\partial}{\partial y}\left(\mu(t,y)M(t,y)\right) = \frac{\partial}{\partial t}\left(\mu(t,y)N(t,y)\right),$$

and multiply the ODE by μ:

$$\underbrace{\mu(t,y(t))M(t,y(t))}_{\widetilde{M}} + \underbrace{\mu(t,y(t))N(t,y(t))}_{\widetilde{N}}\, y'(t) = 0, \qquad \text{with } \frac{\partial \widetilde{M}}{\partial y} = \frac{\partial \widetilde{N}}{\partial t}.$$

Such a function μ is called an *integrating factor* of the ODE. Finding one is in general as hard as solving the ODE itself, but there are tractable special cases: e.g., when $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)/N$ depends on t only, we may take $\mu = \mu(t)$.

2.4 The phase line

Pictures are often more useful than formulas for analyzing nonlinear autonomous ODEs. Accordingly, we will interpret a differential equation as a *vector field*.

Example 2.6. Consider the nonlinear ODE $\dot{x} = \sin x$ with $x(0) = \pi/4$. What are the qualitative features of this ODE? What is $\lim_{t\to\infty} x(t)$?

Let's solve the ODE using separation of variables:

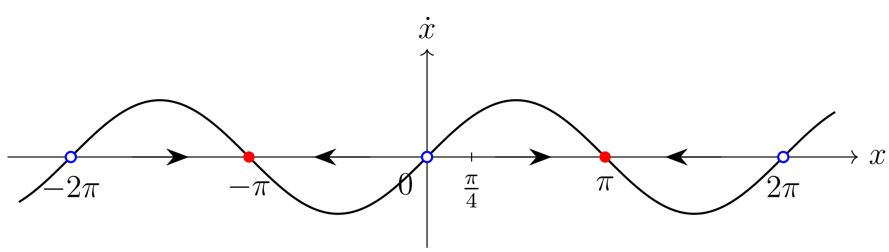
$$\int_{\pi/4}^{x(t)} \frac{1}{\sin z}\, dz = t.$$

The function $z \mapsto -\ln|\csc z + \cot z|$ is an antiderivative of $1/\sin z$, and this leads to

$$t = \ln\left|\frac{\csc(\pi/4) + \cot(\pi/4)}{\csc(x(t)) + \cot(x(t))}\right| \quad \Longrightarrow \quad x(t) = 2\arctan\left(\frac{e^t}{1+\sqrt{2}}\right).$$

We have an exact solution! However, is it helpful? It's not obvious what the limit $\lim_{t\to\infty} x(t)$ is equal to. What can we say qualitatively about the dynamics of the ODE?

Let's instead analyze the ODE geometrically, by plotting $\dot{x}$ against x. Remember that $\dot{x} = \sin(x(t))$, so we are really plotting $\sin x$ versus x.



Let's think of t as time, x as the position of an imaginary particle, and $\dot{x}$ as its velocity. Then the ODE $\dot{x}(t) = \sin(x(t))$ represents a *vector field* on the line: it dictates the velocity $\dot{x}$ at each x. We draw arrows on the plot of $\dot{x}$ versus x to indicate the corresponding velocity

$\dot{x}$ at each x. Here, a right arrow means $\dot{x} > 0$ (the particle moves to the right), and a left arrow means $\dot{x} < 0$ (the particle moves to the left).

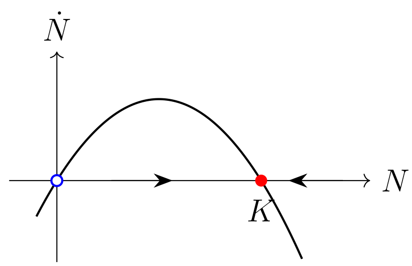
The points at which $\dot{x} = 0$ are called *fixed points* (or *equilibria*). Filled (red) dots denote stable fixed points (attractors or sinks); open (blue) dots denote unstable fixed points (repellers or sources).

So, given $x(0) = \pi/4$, what is $\lim_{t\to\infty} x(t)$? Looking at the graph, we observe that $\lim_{t\to\infty} x(t) = \pi$, and indeed, as $t \to \infty$, the argument $e^t/(1+\sqrt{2})$ in the explicit solution tends to infinity, so $x(t) \to 2 \cdot \pi/2 = \pi$.

Example 2.7 (Population growth and the logistic equation). Recall the logistic equation ODE

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) \qquad \text{(Verhulst, 1838)},$$

where K is the carrying capacity. This ODE has an exact solution (Assignment 3), but let's look at its graphical solution:



We see that, for every initial value $N_0 > 0$, $\lim_{t\to\infty} N(t) = K$, the carrying capacity.

Remark 2.1. This analysis applies only to first-order autonomous ODEs $\dot{x}(t) = f(x(t))$. We will extend the qualitative viewpoint to systems in the plane — phase portraits, stability, and the classification of equilibria — in the chapters on linear and nonlinear systems.

2.5 Existence and uniqueness of solutions

What can we say about the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{4}$$

in general?

- Do solutions exist?
- If so, when is the solution unique?
- If solutions exist, do they exist for all $t \geq t_0$ or only on some interval $[t_0, T]$?

As we will see, these are not trivial questions!

Example 2.8. Consider the IVP

$$y'(t) = \frac{1}{1+y(t)}, \qquad y(0) = -1.$$

This IVP has no solution! More precisely, no function differentiable at $t = 0$ solves it on an interval containing $t = 0$: such a function would have to satisfy the equation at $t = 0$, where it reads $y'(0) = 1/(1 + y(0)) = 1/0$. The right-hand side is undefined at the initial point. We can still "naively" separate variables to find

$$(1 + y(t))y'(t) = 1 \iff \frac{d}{dt}\left(\frac{(1+y(t))^2}{2} - t\right) = 0.$$

With the initial condition, we find that $(1 + y(t))^2 = 2t$, yielding

$$y_\pm(t) = -1 \pm \sqrt{2t}.$$

Both branches solve the ODE for every t and tend to -1 as $t \to 0^+$. But note that $y'_\pm(t) = \pm 1/\sqrt{2t}$ grows without bound as $t \to 0^+$. Thus there is no function differentiable at $t = 0$ that can satisfy the ODE. What is going on? **Note:** the function $f(t, y) = 1/(1 + y)$ is not continuous at $(0, -1)$, and is not even defined there...

Example 2.9. Consider the IVP

$$y'(t) = y(t)^{1/3}, \qquad y(0) = 0.$$

This IVP has a trivial solution, namely $y \equiv 0$. Actually, we can find another one if we separate variables and integrate:

$$\int_{y(0)}^{y(t)} \frac{1}{z^{1/3}}\,dz = \int_0^t 1\,d\tau \quad\iff\quad \frac{3}{2}y(t)^{2/3} = t.$$

Hence $y(t) = \left(\frac{2t}{3}\right)^{3/2}$ is another solution on $t \geq 0$. Therefore, the ODE has more than one solution!

Example 2.10. The solution to $y'(t) = 1 + y(t)^2$ starting at $y(0) = 0$ exists and is unique, but it's defined only for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Separate variables and integrate:

$$\int \frac{1}{1+y^2}\,dy = t + C \quad\iff\quad \arctan(y(t)) = t + C.$$

Evaluating at $t = 0$ gives $C = 0$, so $y(t) = \tan t$. But $y(t) \to +\infty$ as $t \to (\pi/2)^-$ and $y(t) \to -\infty$ as $t \to (-\pi/2)^+$. The solution does not exist for all $t \in \mathbb{R}$, and this asymptotic behavior is called *finite-time blow-up.*

So, what can we say about the *existence* and *uniqueness* of solutions to ODEs? Let's start with existence. Fix $C > 0$ and $T > t_0$, and consider the rectangle

$$D_T = \{(t, y) : t \in [t_0, T] \text{ and } y \in [y_0 - C, y_0 + C]\} \tag{5}$$

centered on the initial point.

Theorem 2.1 (Peano, 1886). *Suppose f is continuous on D_T and set $B = \max_{(t,y)\in D_T} |f(t,y)|$. Then the IVP (4) has at least one continuously differentiable solution on $[t_0, t_0+\alpha]$, where*

$$\alpha = \min\left\{T - t_0, \frac{C}{B}\right\},$$

with the convention $C/0 = \infty$, and the graph of that solution lies in D_T.

Proof. Deferred to later in the course. □

Now, what can we say about *uniqueness*? A classical result is the *Picard–Lindelöf theorem*, which gives a set of sufficient (but not necessary) conditions for a solution to exist and be unique. Suppose that there exist constants $C, K, L > 0$ such that the following three assumptions hold:

(A1) The function f is continuous on the rectangle D_T of (5).

(A2) The function $t \mapsto f(t, y_0)$ is bounded by K for every $t \in [t_0, T]$, i.e.,

$$|f(t,y_0)| \le K \quad \forall\, t \in [t_0, T].$$

(A3) The function $y \mapsto f(t,y)$ is *Lipschitz continuous* on D_T, that is,

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2| \qquad \forall\, t \in [t_0,T],\ \forall\, y_1, y_2 \in [y_0 - C, y_0 + C].$$

Theorem 2.2 (Picard–Lindelöf, 1894). *Suppose assumptions (A1), (A2), and (A3) are satisfied, and that*

$$C \ge \frac{K}{L}\left(e^{L(T-t_0)} - 1\right).$$

Then there exists a unique solution y to the IVP (4) that is continuously differentiable on $[t_0, T]$. Furthermore, $|y(t) - y_0| \le C$ for every $t \in [t_0, T]$; this means that the graph $\{(t, y(t)) : t \in [t_0, T]\}$ lies in D_T.

Proof. Deferred to later in the course. □

Example 2.11. Consider again the IVP $y'(t) = 1 + y(t)^2$ with $y(0) = 0$ from Example 2.10. Here, we have $f(t,y) = 1 + y^2$.

(1) The function f is continuous on every rectangle $D_T = [t_0, T] \times [y_0 - C, y_0 + C]$.

(2) Here, $|f(t, y_0)| = 1 + y_0^2 = 1$ ($\equiv K$), since $y_0 = 0$.

(3) We have

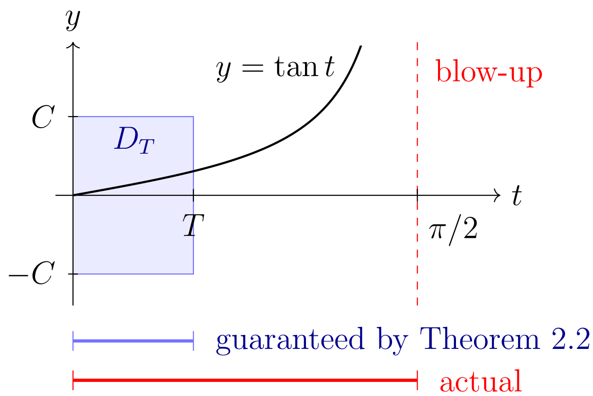
$$\begin{aligned} |f(t,y_1) - f(t,y_2)| = |y_1^2 - y_2^2| = |(y_1 - y_2)(y_1 + y_2)| &= |y_1 + y_2|\,|y_1 - y_2| \\ &\le (|y_1| + |y_2|)\,|y_1 - y_2| \le \underbrace{2C}_{\equiv L}|y_1 - y_2|, \end{aligned}$$

since $y_0 = 0$.

For the theorem, we require

$$C \geq \frac{K}{L}\left(e^{L(T-t_0)} - 1\right) \iff 2C^2 \geq \left(e^{2CT} - 1\right) \iff 0 \leq T \leq \frac{1}{2C}\ln\left(1 + 2C^2\right).$$

This gives $T \leq \frac{1}{4}\ln 9 \approx 0.549$ with $C = 2$. The estimate is conservative: the solution $y(t) = \tan t$ runs until $\pi/2 \approx 1.571$ before it blows up. The picture below compares the interval the theorem guarantees with the interval on which the solution actually exists.



2.6 A first look at numerical approximation

Consider the nonlinear IVP

$$y'(t) = t^2 + y(t)^2, \qquad y(0) = 1. \tag{6}$$

What is $y(1/2)$ equal to? Unfortunately, even though the solution exists at $t = 1/2$ and is unique, it turns out that the ODE cannot be solved in closed form!

Most ODEs arising in applications are more complicated than those we have seen so far, and they are either very hard or impossible to solve in closed form. We must therefore resort to other methods, namely, *numerical methods*.

Recall the definition of the derivative,

$$y'(t) = \lim_{h\to 0}\frac{y(t+h) - y(t)}{h}.$$

When $h > 0$ is small, the quotient on the right is close to $y'(t)$. The ODE says that $y'(t) = f(t, y(t))$, so

$$\frac{y(t+h) - y(t)}{h} \approx f(t, y(t)) \iff y(t+h) \approx y(t) + hf(t, y(t)).$$

Now repeat the step. Fix a step size $h > 0$, set

$$t_n = t_0 + nh, \qquad n = 0, 1, 2, \ldots,$$

and let y_n stand for our approximation of $y(t_n)$. Starting from $y_0 = y(t_0)$ and applying the rule at each t_n gives *Euler's method*,

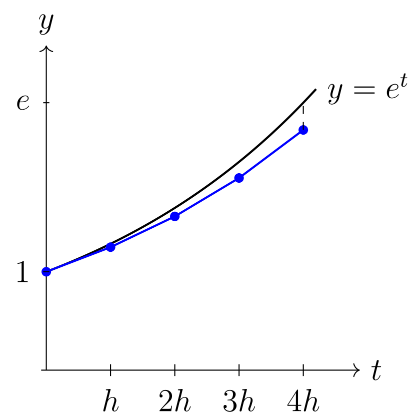
$$\boxed{y_{n+1} = y_n + hf(t_n, y_n), \qquad n = 0, 1, 2, \ldots}$$

The method also has a simple picture. The equation $y' = f(t, y)$ assigns a slope $f(t, y)$ to each point (t, y) of the plane, the slope of the solution curve through that point. From (t_n, y_n) we follow the straight line of slope $f(t_n, y_n)$ for a time h and arrive at (t_{n+1}, y_{n+1}). There we read off the new slope and repeat. What we trace out is a polygon, and each of its segments is tangent at its left endpoint to a solution curve.

Example 2.12. Take $y'(t) = y(t)$ with $y(0) = 1$, whose solution is $y(t) = e^t$. Here $f(t, y) = y$, so each step multiplies by $1 + h$,

$$y_{n+1} = y_n + hy_n = (1 + h)y_n \qquad \Longrightarrow \qquad y_n = (1 + h)^n.$$

Fix $t = 1$ and let $h = 1/N$, so that N steps land on $t_N = 1$. Then $y_N = (1 + 1/N)^N$, which tends to e as N grows. The approximation does converge to the true value, but slowly. With $h = 1/4$ we get $y_4 = (5/4)^4 \approx 2.4414$, off by about 0.28. Halving the step to $h = 1/8$ gives $y_8 = (9/8)^8 \approx 2.5658$, off by about 0.15. Halving h roughly halves the error.



The behavior seen in Example 2.12 follows from *Taylor's theorem*. Suppose y is twice continuously differentiable. Then for each n there is a point ξ_n in (t_n, t_{n+1}) with

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}y''(\xi_n).$$

One Euler step taken from the exact value $y(t_n)$ is therefore off by an amount of order h^2, called the *local truncation error*. Reaching a fixed time T takes $N = (T - t_0)/h$ steps, so we expect a total error of order $Nh^2 = (T - t_0)h$, one power of h smaller. The count is crude, since it ignores how earlier errors grow as later steps carry them along. The Lipschitz condition (A3) keeps that growth in check. More precisely:

Proposition 2.1 (First-order convergence of Euler's method)**.** *Suppose that assumptions (A1)–(A3) hold, that the exact solution y of* (4) *satisfies $|y''(t)| \le M$ for all $t \in [t_0, T]$, and that y_n is generated by Euler's method with step size $h = (T - t_0)/N$. Suppose furthermore that $|y_n - y_0| \le C$ for every $n \in \{0, 1, \ldots, N\}$, so that the Lipschitz condition (A3) applies at the computed values as well. Then*

$$\max_{0 \le n \le N} |y(t_n) - y_n| \le \frac{Mh}{2L}\left(e^{L(T - t_0)} - 1\right).$$

Proof. Deferred to later in the course. $\square$

The error shrinks linearly in the step size. For this reason, we call Euler's method a method of *order one*. We will prove the proposition and investigate other methods later.

Example 2.13. We return to the IVP (6) that opened the section, with $f(t, y) = t^2 + y^2$ and $y_0 = 1$. Two steps of size $h = 1/4$ reach $t = 1/2$ and cost two evaluations of f,

$$y_1 = 1 + \tfrac{1}{4}\left(0^2 + 1^2\right) = 1.25,$$
$$y_2 = 1.25 + \tfrac{1}{4}\left((1/4)^2 + (1.25)^2\right) = 1.65625,$$

so that $y(1/2) \approx 1.65625$. Halving the step size repeatedly gives the table below, where we use as a reference the value $y(1/2) = 2.06700$, accurate to five decimal places.

h	$y_N \approx y(1/2)$	error
1/4	1.65625	0.41075
1/8	1.79803	0.26897
1/16	1.90641	0.16059
1/32	1.97770	0.08930
1/64	2.01963	0.04737
1/128	2.04256	0.02444

The ratio of successive errors approaches 1/2, as the first-order bound of Proposition 2.1 leads us to expect.

Summary.

1) Separable equations are solved by separation of variables; the constant solutions $y \equiv y_*$ with $g(y_*) = 0$ must be accounted for separately.

2) Linear first-order inhomogeneous equations are solved by variation of constants; the general solution is the boxed formula of Section 2.2.

3) Exact equations are solved by finding a potential function;

4) For autonomous equations, the phase line reveals the qualitative behavior — fixed points, stability, and long-time limits — without solving the equation.

5) The Picard–Lindelöf theorem guarantees local existence and uniqueness under a Lipschitz condition. Even when the solution is unique, it can still blow up in finite time.

6) Euler's method gives a first computable approximation to the solution of an IVP.