MATH 441: Differential Equations
Exercises for Chapter 3

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1 Exercises (Second-order linear ODEs)

Exercise 1.1. a) Find the dimension of $\mathrm{Span}(\{\sin^2 t, \cos^2 t, \cos(2t), 1\})$ and give a basis.
b) Are the functions e^t, e^{-t}, and $\cosh t$ linearly independent on $\mathbb{R}$? Justify your answer with Lemma 3.1 of the notes (the test with coefficients $\alpha_1, \dots, \alpha_n$).
c) Show that $\cosh t$ and $\sinh t$ form a fundamental set of solutions of

$$\frac{d^2y}{dt^2} - y = 0,$$

and find the solution that satisfies $y(0) = 2$ and $y'(0) = -1$.

Solutions:

Solutions (continued):

Exercise 1.2. Let $y_1(t) = t^2$ for $t \in \mathbb{R}$, and let

$$y_2(t) = \begin{cases} t^2 & \text{if } t \geq 0, \\ -t^2 & \text{if } t < 0. \end{cases}$$

a) Show that y_2 is differentiable on $\mathbb{R}$. Compute the Wronskian $W[y_1, y_2](t)$ for every $t \in \mathbb{R}$.
b) Show that y_1 and y_2 are linearly independent on $\mathbb{R}$.
c) Explain why a) and b) do not contradict the Wronskian test for solutions. Deduce that y_1 and y_2 cannot both solve an equation $y'' + a_1(t)y' + a_0(t)y = 0$ whose coefficients are continuous on an open interval containing $t = 0$.

Solutions:

Solutions (continued):

Exercise 1.3. Consider the ODE

$$t^2 \frac{d^2y}{dt^2} - t(t+2)\frac{dy}{dt} + (t+2)y = 0, \qquad t > 0.$$

a) Verify that the function $t \mapsto y_1(t) = t$ solves the ODE.
b) Use Abel's identity to find the Wronskian of y_1 and any second solution y_2, up to a multiplicative constant.
c) Show that $(y_2/y_1)' = W[y_1, y_2]/y_1^2$, and use this identity to find a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental set of solutions.

Solutions:

Solutions (continued):

Exercise 1.4. Solve the following initial value problems.

a) $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 0$, with $y(0) = 1$ and $y'(0) = -2$.

b) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 0$, with $y(0) = 1$ and $y'(0) = 0$.

c) $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 9y = 0$, with $y(0) = 2$ and $y'(0) = 1$.

Solutions:

Solutions (continued):

Exercise 1.5. Let $\omega_0 > 0$. The complex-valued functions $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ solve the harmonic oscillator $y'' + \omega_0^2 y = 0$, and so does every combination $y(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$ with $C_1, C_2 \in \mathbb{C}$.

a) Show that $y(t)$ is real for every $t \in \mathbb{R}$ if and only if $C_2 = \overline{C_1}$.
b) Suppose that $C_2 = \overline{C_1}$. Show that $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ with $c_1 = 2 \operatorname{Re} C_1$ and $c_2 = -2 \operatorname{Im} C_1$.

Solutions:

Solutions (continued):

Exercise 1.6. Find the general solution of each ODE.

a) $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} = 0.$

b) $\dfrac{d^3y}{dt^3} + y = 0.$

c) $\dfrac{d^4y}{dt^4} + 2\dfrac{d^2y}{dt^2} + y = 0.$

Consider now the ODE $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} = 0.$

d) Solve the initial value problem with $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 3$.
e) Solve the boundary value problem with $y(0) = 0$, $y(1) = 2$, and $y'(0) = 0$.

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.7. The following table refines the method of undetermined coefficients for the equation $y'' + a_1 y' + a_0 y = f(t)$ with constant coefficients. In each row, m is the multiplicity of the number 0, r, $i\omega$, or r, respectively, as a root of the characteristic polynomial p, with $m = 0$ when that number is not a root.

$f(t)$	Form of the particular solution
$b_0 + b_1 t + \cdots + b_d t^d$	$t^m \left(A_0 + A_1 t + \cdots + A_d t^d\right)$
e^{rt}	$t^m C e^{rt}$
$\cos(\omega t)$ or $\sin(\omega t)$	$t^m \left(A\cos(\omega t) + B\sin(\omega t)\right)$
$\left(b_0 + b_1 t + \cdots + b_d t^d\right) e^{rt}$	$t^m \left(A_0 + A_1 t + \cdots + A_d t^d\right) e^{rt}$

Find a particular solution of each ODE.

a) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 4t^2.$

b) $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = e^t.$

c) $\dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = e^t.$ (The method extends to order n.)

Solutions:

Solutions (continued):

Solutions (continued):

Exercise 1.8. Consider the ODE

$$\frac{d^2y}{dt^2} + y = \tan(t), \qquad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Use variation of constants to find a particular solution, and then the general solution. (You will need the antiderivative $\int \sec(t)\, dt = \ln\left(\sec(t) + \tan(t)\right)$ on $(-\pi/2, \pi/2)$.)

Solutions:

Solutions (continued):

Exercise 1.9. Let $\omega_0 > 0$, let f be continuous on $\mathbb{R}$, and consider $y'' + \omega_0^2 y = f(t)$.

a) Use the fundamental set $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$ in the variation-of-constants formula to show that

$$y_p(t) = \frac{1}{\omega_0} \int_{t_0}^{t} \sin\left(\omega_0 (t - \tau)\right) f(\tau)\, d\tau.$$

b) Show that $y_p(t_0) = 0$ and $y_p'(t_0) = 0$. (Hint: $\frac{d}{dt} \int_{t_0}^{t} K(t,\tau)\, d\tau = K(t,t) + \int_{t_0}^{t} \frac{\partial K}{\partial t}(t,\tau)\, d\tau$ for continuously differentiable K.)

c) Take $t_0 = 0$ and $f(t) = \cos(\omega_0 t)$. Evaluate the integral in a) and compare the result with the resonant solution in Section 3.5 of the notes.

Solutions:

Solutions (continued):

Exercise 1.10. Consider the damped oscillator $my'' + \gamma y' + ky = 0$ with $m = 1$ and $k = 4$.

a) For which values of $\gamma > 0$ is the oscillator overdamped, critically damped, or underdamped?
b) Let $\gamma = 4$, $y(0) = 1$, and $y'(0) = v_0$. Find the solution, and show that the mass passes through the equilibrium position at some time $t > 0$ if and only if $v_0 < -2$.
c) Let $\gamma = 2$. Find the damped frequency ω_d and the solution with $y(0) = 1$ and $y'(0) = 0$.

Solutions:

Solutions (continued):