MATH 441: Assignment 3

Instructions: Submit your *handwritten* solutions in class by *Friday, October 2nd, 11 AM.* I will *not* accept typed solutions unless you have an accommodation.

Readings

Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima.

Boyce and DiPrima: Sections 2.4–2.8 and 3.2

Problem 1

Consider the initial value problem

$$\left(3t^2y(t) + 2ty(t) + y(t)^3\right) + \left(t^2 + y(t)^2\right) y'(t) = 0, \quad y(0) = 1.$$

Write $M(t,y) = 3t^2y + 2ty + y^3$ and $N(t,y) = t^2 + y^2$.

a) Show that the equation is not exact.

b) Let $\mu = \mu(t)$ be a nonvanishing differentiable function of t alone. Show that, at points where $N(t,y) \neq 0$, the equation $\mu M + \mu N y' = 0$ is exact if and only if

$$\mu'(t) = \mu(t) \frac{\frac{\partial M}{\partial y}(t,y) - \frac{\partial N}{\partial t}(t,y)}{N(t,y)}.$$

Verify that the quotient depends on t only, and find an integrating factor $\mu(t)$.

c) Find a potential function V for the exact equation obtained in b), and solve the initial value problem in implicit form.

Problem 2

Let $r > 0$ and $K > 0$.

a) Solve the logistic equation

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right), \quad N(0) = N_0 > 0.$$

Hint: You can either solve it directly using partial fractions or make the change of variable $x(t) = 1/N(t)$ and then derive and solve the corresponding ODE for $x(t)$.

b) Compute $\lim_{t\to+\infty} N(t)$ from your solution, and check that the answer agrees with the phase line of the logistic equation in the notes.

Suppose now that the population is harvested at a constant rate $H > 0$:

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) - H.$$

c) Suppose that $0 < H < rK/4$. Find the two equilibria $N_- < N_+$, draw the phase line, and classify each equilibrium as stable or unstable. What happens to the population if $N_0 > N_-$? If $0 < N_0 < N_-$?

d) Suppose that $H > rK/4$. Show that $N'(t) \le rK/4 - H < 0$ as long as $N(t) \ge 0$, and conclude that the population reaches 0 in finite time, whatever the value of $N_0 > 0$.

Problem 3

The proof of the Picard–Lindelöf theorem (Theorem 2.2 in the notes) is constructive. You'll explore its main idea in this problem. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

with f continuous. Starting from the constant function $\varphi_0(t) = y_0$, define the *Picard iterates*

$$\varphi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, \varphi_k(\tau))\, d\tau, \qquad k = 0, 1, 2, \ldots$$

Under assumptions (A1)–(A3) in Theorem 2.2 of the notes, the iterates converge on $[t_0, T]$ to the unique solution of the initial value problem. We will prove this later in the course.

a) Show that a continuously differentiable function y solves the initial value problem on $[t_0, T]$ if and only if it satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau \quad \text{for every } t \in [t_0, T].$$

(Thus a solution is a fixed point of the map $\varphi \mapsto y_0 + \int_{t_0}^{t} f(\tau, \varphi(\tau))\, d\tau$.)

Now consider the initial value problem

$$y'(t) = 1 - 2ty(t), \quad y(0) = 0.$$

b) Let $T = 1/2$. Show that assumptions (A1), (A2), and (A3) hold on the rectangle D_T for every $C > 0$, with $K = 1$ and $L = 1$. Choose C so that Theorem 2.2 guarantees a unique solution on $[0, 1/2]$.

c) Compute the Picard iterates φ_1, φ_2, and φ_3.

d) Use variation of constants to show that the solution is

$$y(t) = e^{-t^2} \int_0^t e^{s^2}\, ds.$$

(Do not attempt to evaluate this integral.)

e) Expand e^{s^2} and e^{-t^2} in Taylor series to show that the Taylor polynomial of degree 5 of the solution in d) equals $\varphi_3(t)$.

Problem 4

Let $t > 0$ and $N \in \mathbb{N}$, and apply Euler's method with step size $h = t/N$ to the initial value problem

$$y'(s) = y(s), \quad y(0) = 1,$$

whose solution is $y(s) = e^s$.

a) Show that after N steps, Euler's method gives $y_N = (1 + t/N)^N$.

b) Using the inequalities $x - \frac{x^2}{2} \leq \ln(1 + x) \leq x$ for every $x \geq 0$, deduce that

$$0 \leq e^t - \left(1 + \frac{t}{N}\right)^N \leq \frac{t^2 e^t}{2N}.$$

Hint: Write $(1 + t/N)^N = e^{N \ln(1+t/N)}$, and use $e^{-u} \geq 1 - u$ for $u \geq 0$.

c) Rewrite the bound in b) in terms of h, and explain why it shows that Euler's method has order one.

Problem 5

Let $\lambda > 0$, and consider the initial value problem

$$y'(t) = -\lambda y(t), \quad y(0) = y_0 \neq 0.$$

a) Solve the problem, and compute $\lim_{t\to\infty} y(t)$.

b) Apply Euler's method with step size $h > 0$, and find a formula for y_n. For which values of h does the sequence $\{y_n\}_{n=0}^{\infty}$ converge to 0? For which values of h is it bounded? For which values of h do all the y_n have the same sign as y_0?

c) The *implicit* (or backward) Euler method evaluates f at the new point instead of the old one:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

Apply it to this problem, solve for y_{n+1}, and show that $y_n \to 0$ as $n \to \infty$ for every step size $h > 0$.

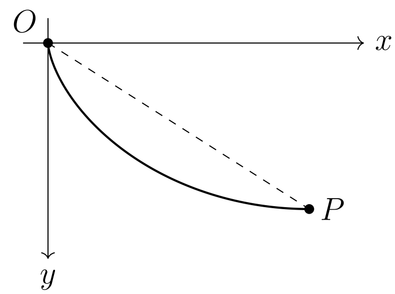
Note: Parts b) and c) illustrate the *stability* of numerical methods, which we will study in the chapter on numerical methods.

Problem 6 (The brachistochrone)

In 1696, Johann Bernoulli asked for the curve along which a bead, sliding without friction under gravity, travels from a point O to a lower point P in the shortest time. This curve is called the *brachistochrone*, from the Greek for "shortest time." Place O at the origin, measure x to the right and y *downward*, and describe the curve as the graph of a function $y = y(x)$ with $y(0) = 0$. Bernoulli showed that the brachistochrone satisfies the ODE

$$y(x)\left(1 + y'(x)^2\right) = k^2$$

for some constant $k > 0$. You may take this fact as given.



a) Show that $0 < y(x) \le k^2$ at every point x where the ODE holds, and that on an interval where $y' > 0$, the ODE is equivalent to the separable equation

$$y'(x) = \sqrt{\frac{k^2 - y(x)}{y(x)}}.$$

b) Separate the variables and use the substitution $y = k^2 \sin^2\phi$, with $0 \le \phi \le \pi/2$, to show that the solution with $y(0) = 0$ is given in parametric form by

$$x = k^2\left(\phi - \sin\phi\cos\phi\right), \qquad y = k^2\sin^2\phi.$$

Hint: $\sin^2\phi = \frac{1}{2}\left(1 - \cos(2\phi)\right)$.

c) Set $\theta = 2\phi$ and $a = k^2/2$. Show that the curve in b) is the *cycloid*

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta),$$

the curve traced by a point on a circle of radius a that rolls along the x-axis. Check directly that $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$ along the cycloid, and that the cycloid satisfies the ODE $y(1 + y'^2) = k^2$.

d) Let $P = (\pi a, 2a)$, which corresponds to $\theta = \pi$. The travel time from O to P along a curve $y = y(x)$ is proportional to the (improper, but convergent) integral

$$J[y] = \int_0^{\pi a} \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}\,dx.$$

(You may take this fact as given.) Show that $J = \pi\sqrt{2a}$ along the cycloid and $J = \sqrt{\pi^2 + 4}\sqrt{2a}$ along the straight line from O to P (dashed in the figure). Which curve is faster? **Hint:** Along the cycloid, write the integral in terms of θ, using $\sqrt{1 + y'(x)^2}\,dx = \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta$ and $1 - \cos\theta = 2\sin^2(\theta/2)$.

Problem 7

Consider the ODE

$$ty''(t) - (t+1)y'(t) + y(t) = 0.$$

a) Verify that $y_1(t) = 1 + t$ and $y_2(t) = e^t$ are solutions for every $t \in \mathbb{R}$.

b) Compute the Wronskian $W[y_1, y_2](t)$, and show that $\{y_1, y_2\}$ is a fundamental set of solutions on $I = (0, \infty)$.

c) Write the ODE in the form $y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$ on $I = (0, \infty)$, and check that your Wronskian satisfies Abel's identity.

d) The Wronskian vanishes at $t = 0$ but at no other point. Explain why this does not contradict Corollary 3.1 in the notes, which states that the Wronskian of two solutions vanishes either everywhere or nowhere.

e) Find the solution on $(0, \infty)$ that satisfies $y(1) = 0$ and $y'(1) = 1$.

Problem 8

Let I be an open interval, and let a_1 and a_0 be continuous on I.

a) Let y_1 and y_2 solve $y'' + a_1(t)y' + a_0(t)y = 0$ on I. Show that if y_1 and y_2 vanish at the same point $t_* \in I$, then $\{y_1, y_2\}$ is not a fundamental set of solutions.

b) Show the same conclusion if y_1 and y_2 both have a local maximum at the same point $t_* \in I$.

c) Conversely, let y_1 and y_2 be twice continuously differentiable functions on I whose Wronskian $W = W[y_1, y_2]$ does not vanish at any point of I. Show that y_1 and y_2 both solve exactly one equation of the form

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

with continuous coefficients on I, and that its coefficients are

$$a_1 = -\frac{W'}{W}, \qquad a_0 = \frac{y_1'y_2'' - y_1''y_2'}{W}.$$

Hint: The two conditions $y_j'' + a_1y_j' + a_0y_j = 0$, $j = 1, 2$, form a linear system of two equations for the two unknowns $a_1(t)$ and $a_0(t)$.

d) Apply c) to $y_1(t) = t$ and $y_2(t) = t^2$ on $I = (0, \infty)$.