MATH 441: Homework 2

Instructions: Submit your *handwritten* solutions in class by *Friday, September 18th, 11 AM.* I will *not* accept typed solutions unless you have an accommodation.

Readings

Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima. Previous editions are just as good; just find the corresponding pages and materials.

Boyce and DiPrima: Chapter 1 and Sections 2.1–2.3

Problem 1

Consider the Bernoulli equation

$$y'(t) + f(t)y(t) = g(t)y(t)^n.$$

where $n \in \mathbb{R} \setminus \{0, 1\}$. Using the coordinate transformation $\varphi(t) = y(t)^q$ and a suitable choice of q, turn the Bernoulli equation into a linear equation. (Do not solve the resulting ODE.)

Problem 2

For which $\lambda > 0$ does the following ODE have a unique solution?

$$y'(t) - y(t)^\lambda = 0, \ y(0) = 0, \ \text{with } t \geqslant 0$$

Problem 3

Consider the initial value problem (due to Augustin-Louis Cauchy, 1824):

$$(y'(t))^3 = 8y(t)^2 - 4ty(t)y'(t), \ y(0) = y_0, \tag{1}$$

where $y_0 \geqslant 0$ is a given initial value. Show there exists a constant $C \in \mathbb{R}$ such that $y(t) = C(t + C)^2$ is a solution to the problem above. For $y_0 = 0$, find yet another solution of (1), that is, show the solution is not unique.

Problem 4

Consider the IVP

$$ty'(t) + \lambda y(t) = g(t), \; y(0) = y_0,$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and $\lambda \geqslant 0$, $y_0 \in \mathbb{R}$. Show that the solution to this IVP, if it exists, is unique for $t \geqslant 0$. **Hint:** Suppose there are two solutions y_1, y_2 and define $w = y_1 - y_2$. Prove that $w \equiv 0$, that is, $w(t) = 0$ identically on $t \geqslant 0$.

Note: The hint given in this problem is a standard technique for proving (or disproving) uniqueness of solutions to ODEs (and solutions to other types of differential equations).

Problem 5

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, with at least one zero in $\mathbb{R}$, and let $y_0 \in \mathbb{R}$. Consider the implicit ODE problem

$$y'(t)f'(y(t)) + f(y(t)) = 0, \; y(0) = y_0.$$

Assume there exists a sufficiently smooth and bounded solution $t \mapsto y(t)$ to the problem above, with $f(y(t)) \neq 0$ over $t \in [0, \infty)$. Prove that

$$f(\lim_{t\to\infty} y(t)) = 0.$$

Hint: You may find it helpful to use the chain rule: $(f(y(t)))' = y'(t)f'(y(t))$.

Problem 6

Find the implicit solution of the equation (due to Gottfried Wilhelm Leibniz):

$$y'(t) = -\frac{y(t)}{\sqrt{a^2 - y(t)^2}}, \; a > 0.$$

Hint: Use the substitution $z^2 = a^2 - y^2$, $z\,dz = -y\,dy$, in the integration.

Problem 7

Consider the nonlinear system of ODEs for an unknown (vector-valued) function $t \mapsto \boldsymbol{y}(t) = (y_1(t), y_2(t), y_3(t))$:

$$\begin{aligned} y_1'(t) &= -0.04y_1(t) + 10^4 y_2(t)y_3(t), \\ y_2'(t) &= 0.04y_1(t) - 10^4 y_2(t)y_3(t) - 10^7 y_2(t)^2, \; \boldsymbol{y}(0) = \boldsymbol{y}_0 \\ y_3'(t) &= 10^7 y_2(t)^2. \end{aligned}$$

a) Show that the sum $y_1(t)+y_2(t)+y_3(t)$ of the three solution components is independent of t for $t \geqslant 0$. **Hint:** Do not attempt to solve this system.

b) Prove that, for initial values $\boldsymbol{y}_0 \in [0, \infty)^3$, the solution $t \mapsto \boldsymbol{y}(t)$ remains in $[0, \infty)^3$ for every $t \geqslant 0$ for which the solution exists.

Problem 8

Compute the solution to each of the following separable ODEs:

a) $y'(t) + t\sin(t) = 0$.

b) $t + y(t)y'(t) = 0$.

c) $\sinh y(t)y'(t) = te^t$.

d) $y'(t) = ty(t)(1 - y(t))$.

e) $y'(t) - y(t)^2 = 1$.

Problem 9

Compute the solution to each of the following linear inhomogeneous ODEs using the *variation of constants* technique:

a) $y'(t) + 2y(t) = e^{-2t} + t$.

b) $y'(t) + \dfrac{y(t)}{t} = \cos(t)$, where $t > 0$.

c) $y'(t) + y(t)\tan(t) = \sin(2t)$, where $t \in (-\pi/2, \pi/2)$.

"Problem" 10

I would like to learn more about you. Please answer the following questions. If you don't have an answer yet for any of these, just write N/A or None.

a) Are you an undergraduate or graduate student? At what stage are you in your studies? What is your major or intended major?

b) What would you like to achieve by the end of this course?

c) What would you like to achieve by the end of your studies and career-wise (if you already know or have an idea)?

d) Do you have any questions for me, the instructor?

Note: Ordinary differential equations is a vast and important subject that finds applications across mathematics, science and engineering. This includes traditional fields such as geometry, physics, economics, biology and engineering, and more "trendy" fields such as machine learning, deep learning, and data science.

I intend to cover applications across these fields, but depending on your interests I can adjust accordingly and design applied problems in your field(s) of interest.