

MATH 441: Solutions to Assignment 2

Problem 1

Consider the Bernoulli equation

$$y'(t) + f(t)y(t) = g(t)y(t)^n.$$

where $n \in \mathbb{R} \setminus \{0, 1\}$. Using the coordinate transformation $\varphi(t) = y(t)^q$ and a suitable choice of q , turn the Bernoulli equation into a linear equation. (Do not solve the resulting ODE.)

Solution: Suppose that y solves the Bernoulli equation on an interval I and that $y(t) > 0$ for every $t \in I$, so that every real power of y is defined and differentiable on I . Let $q \neq 0$ and $\varphi(t) = y(t)^q$. The chain rule and the Bernoulli equation give

$$\varphi'(t) = qy(t)^{q-1}y'(t) = qy(t)^{q-1}(g(t)y(t)^n - f(t)y(t)) = qg(t)y(t)^{q+n-1} - qf(t)\varphi(t).$$

The exponent $q + n - 1$ vanishes when $q = 1 - n$, and this choice satisfies $q \neq 0$ because $n \neq 1$. With $q = 1 - n$, the function $\varphi(t) = y(t)^{1-n}$ therefore satisfies

$$\varphi'(t) + (1 - n)f(t)\varphi(t) = (1 - n)g(t),$$

which is a linear inhomogeneous ODE for φ .

Problem 2

For which $\lambda > 0$ does the following ODE have a unique solution?

$$y'(t) - y(t)^\lambda = 0, \quad y(0) = 0, \quad \text{with } t \geq 0$$

Solution: The solution is unique if and only if $\lambda \geq 1$. The power y^λ is defined for every $\lambda > 0$ only when $y \geq 0$. By a solution we therefore mean a differentiable function $y: [0, T) \rightarrow [0, \infty)$, with $0 < T \leq \infty$, that satisfies $y'(t) = y(t)^\lambda$ on $[0, T)$ and $y(0) = 0$. For every $\lambda > 0$, the zero function $y \equiv 0$ is a solution. We ask whether there are others.

Case 1: $0 < \lambda < 1$. Let $p = 1/(1 - \lambda) > 1$ and

$$y(t) = ((1 - \lambda)t)^p, \quad t \geq 0.$$

Then $y(0) = 0$, and, since $p > 1$, the function y is differentiable on $[0, \infty)$ with

$$y'(t) = p(1 - \lambda) ((1 - \lambda)t)^{p-1} = ((1 - \lambda)t)^{p-1}.$$

Since $p - 1 = \lambda/(1 - \lambda) = \lambda p$, we find $y'(t) = ((1 - \lambda)t)^{\lambda p} = y(t)^\lambda$. Hence y is a solution, and y differs from the zero function. The solution is not unique.

Case 2: $\lambda \geq 1$. We show that the zero function is the only solution. Suppose, for a contradiction, that a solution y satisfies $y(t_1) > 0$ for some $t_1 \in (0, T)$. Let

$$t_* = \sup\{t \in [0, t_1] : y(t) = 0\}.$$

The set on the right contains 0, and it is closed because y is continuous. Hence $y(t_*) = 0$, $t_* < t_1$, and $y(t) > 0$ for every $t \in (t_*, t_1]$. On the interval $(t_*, t_1]$, we may divide the ODE by $y(t)^\lambda$ and find $y'(t)/y(t)^\lambda = 1$.

If $\lambda = 1$, then $(\ln y(t))' = 1$ on $(t_*, t_1]$, and integrating from $s \in (t_*, t_1)$ to t_1 gives

$$\ln y(t_1) - \ln y(s) = t_1 - s \leq t_1.$$

As $s \rightarrow t_*^+$, we have $y(s) \rightarrow y(t_*) = 0$, so the left-hand side tends to $+\infty$, while the right-hand side is bounded by t_1 . This is a contradiction.

If $\lambda > 1$, then $(y(t)^{1-\lambda})' = (1 - \lambda)y(t)^{-\lambda}y'(t) = 1 - \lambda$ on $(t_*, t_1]$, and integrating from $s \in (t_*, t_1)$ to t_1 gives

$$y(s)^{1-\lambda} = y(t_1)^{1-\lambda} + (\lambda - 1)(t_1 - s) \leq y(t_1)^{1-\lambda} + (\lambda - 1)t_1.$$

As $s \rightarrow t_*^+$, we have $y(s) \rightarrow 0^+$ and $1 - \lambda < 0$, so the left-hand side tends to $+\infty$, while the right-hand side is bounded. This is a contradiction.

In both subcases, $y(t) = 0$ for every $t \in [0, T)$, so the solution is unique.

Problem 3

Consider the initial value problem (due to Augustin-Louis Cauchy, 1824):

$$(y'(t))^3 = 8y(t)^2 - 4ty(t)y'(t), \quad y(0) = y_0, \tag{1}$$

where $y_0 \geq 0$ is a given initial value. Show there exists a constant $C \in \mathbb{R}$ such that $y(t) = C(t + C)^2$ is a solution to the problem above. For $y_0 = 0$, find yet another solution of (1), that is, show the solution is not unique.

Solution: Let $C \in \mathbb{R}$ and $y(t) = C(t + C)^2$, so that $y'(t) = 2C(t + C)$. The left-hand side of the ODE equals

$$(y'(t))^3 = 8C^3(t + C)^3,$$

and the right-hand side equals

$$8C^2(t + C)^4 - 4t \cdot C(t + C)^2 \cdot 2C(t + C) = 8C^2(t + C)^3 ((t + C) - t) = 8C^3(t + C)^3.$$

Hence y satisfies the ODE for every $C \in \mathbb{R}$. The initial condition reads $y(0) = C^3 = y_0$, which holds for $C = y_0^{1/3}$. Therefore

$$y(t) = y_0^{1/3} \left(t + y_0^{1/3} \right)^2$$

solves the initial value problem (1).

Suppose now that $y_0 = 0$. Then $C = 0$, and the solution above is the zero function. To find another solution, we try $y(t) = At^m$ with $A \neq 0$ and $m > 0$, which satisfies $y(0) = 0$. Then $y'(t) = Amt^{m-1}$ and

$$(y'(t))^3 = A^3 m^3 t^{3m-3}, \quad 8y(t)^2 - 4ty(t)y'(t) = A^2(8 - 4m)t^{2m}.$$

The powers of t agree when $3m - 3 = 2m$, that is, $m = 3$. The coefficients then agree when $27A^3 = -4A^2$, that is, $A = -4/27$. Hence

$$y(t) = -\frac{4}{27}t^3$$

is a second solution of (1) with $y_0 = 0$. As a check, $y'(t) = -4t^2/9$, so $(y'(t))^3 = -64t^6/729$, and

$$8y(t)^2 - 4ty(t)y'(t) = \frac{128}{729}t^6 - \frac{64}{243}t^6 = \frac{128 - 192}{729}t^6 = -\frac{64}{729}t^6.$$

The solution of (1) with $y_0 = 0$ is therefore not unique.

Problem 4

Consider the IVP

$$ty'(t) + \lambda y(t) = g(t), \quad y(0) = y_0,$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function and $\lambda \geq 0$, $y_0 \in \mathbb{R}$. Show that the solution to this IVP, if it exists, is unique for $t \geq 0$. **Hint:** Suppose there are two solutions y_1, y_2 and define $w = y_1 - y_2$. Prove that $w \equiv 0$, that is, $w(t) = 0$ identically on $t \geq 0$.

Solution: By a solution we mean a function that is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, satisfies the ODE for $t > 0$, and takes the value y_0 at $t = 0$. Suppose that y_1 and y_2 are two solutions, and let $w = y_1 - y_2$. Subtracting the ODE for y_2 from the ODE for y_1 , we find

$$tw'(t) + \lambda w(t) = g(t) - g(t) = 0 \quad \text{for every } t > 0, \quad w(0) = y_0 - y_0 = 0.$$

For $t > 0$, the product rule gives

$$(t^\lambda w(t))' = t^\lambda w'(t) + \lambda t^{\lambda-1} w(t) = t^{\lambda-1} (tw'(t) + \lambda w(t)) = 0.$$

Hence there exists a constant $L \in \mathbb{R}$ such that $t^\lambda w(t) = L$ for every $t > 0$. We now let $t \rightarrow 0^+$. Since $\lambda \geq 0$, we have $0 < t^\lambda \leq 1$ for $t \in (0, 1]$, and since w is continuous with $w(0) = 0$,

$$|L| = \lim_{t \rightarrow 0^+} t^\lambda |w(t)| \leq \lim_{t \rightarrow 0^+} |w(t)| = 0.$$

Therefore $L = 0$, and since $t^\lambda > 0$ for $t > 0$, we find $w(t) = 0$ for every $t > 0$. Together with $w(0) = 0$, this shows that $w \equiv 0$ on $[0, \infty)$, that is, $y_1 = y_2$.

Problem 5

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function, with at least one zero in $\mathbb{R}$, and let $y_0 \in \mathbb{R}$. Consider the implicit ODE problem

$$y'(t)f'(y(t)) + f(y(t)) = 0, \quad y(0) = y_0.$$

Assume there exists a sufficiently smooth and bounded solution $t \mapsto y(t)$ to the problem above, with $f(y(t)) \neq 0$ over $t \in [0, \infty)$. Prove that

$$f(\lim_{t \rightarrow \infty} y(t)) = 0.$$

Hint: You may find it helpful to use the chain rule: $(f(y(t)))' = y'(t)f'(y(t))$.

Solution: We take “sufficiently smooth” to mean that y is continuously differentiable on $[0, \infty)$. We proceed in three steps.

Step 1: We compute $f(y(t))$. Let $h(t) = f(y(t))$. By the chain rule and the ODE,

$$h'(t) = y'(t)f'(y(t)) = -f(y(t)) = -h(t).$$

Hence $(e^t h(t))' = e^t (h'(t) + h(t)) = 0$, so $e^t h(t) = h(0) = f(y_0)$ and

$$f(y(t)) = f(y_0)e^{-t} \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Step 2: We show that $\lim_{t \rightarrow \infty} y(t)$ exists. The ODE and the assumption $f(y(t)) \neq 0$ give $y'(t)f'(y(t)) = -f(y(t)) \neq 0$, so $y'(t) \neq 0$ for every $t \geq 0$. Since y' is continuous and never vanishes, the intermediate value theorem shows that y' has a constant sign on $[0, \infty)$. Hence y is strictly monotone. Since y is also bounded, the limit

$$y_\infty = \lim_{t \rightarrow \infty} y(t)$$

exists and is finite.

Step 3: We pass to the limit. Since f is continuous at y_∞ , Steps 1 and 2 give

$$f\left(\lim_{t \rightarrow \infty} y(t)\right) = f(y_\infty) = \lim_{t \rightarrow \infty} f(y(t)) = 0.$$

Problem 6

Find the implicit solution of the equation (due to Gottfried Wilhelm Leibniz):

$$y'(t) = -\frac{y(t)}{\sqrt{a^2 - y(t)^2}}, \quad a > 0.$$

Hint: Use the substitution $z^2 = a^2 - y^2$, $z dz = -y dy$, in the integration.

Solution: The right-hand side of the ODE is defined for $|y| < a$. The zero function $y \equiv 0$ is a solution. Suppose now that y is a solution with $0 < |y(t)| < a$ on an interval. We separate the variables and integrate:

$$-\frac{\sqrt{a^2 - y(t)^2}}{y(t)} y'(t) = 1 \implies \int \frac{-\sqrt{a^2 - y^2}}{y} dy = t + C.$$

The integral on the left is the integral of Problem 4(i) of Assignment 1. We recall how to compute it. Following the hint, we substitute $z = \sqrt{a^2 - y^2}$, so that $0 < z < a$, $z^2 = a^2 - y^2$, and $z dz = -y dy$. Then $dy = -(z/y) dz$ and

$$\int \frac{-\sqrt{a^2 - y^2}}{y} dy = \int \frac{z^2}{y^2} dz = \int \frac{z^2}{a^2 - z^2} dz = \int \left(\frac{a}{2} \left(\frac{1}{a - z} + \frac{1}{a + z} \right) - 1 \right) dz.$$

Since $0 < z < a$, both $a - z$ and $a + z$ are positive, and the last integral equals

$$\frac{a}{2} \ln \left(\frac{a + z}{a - z} \right) - z = a \ln \left(\frac{a + z}{|y|} \right) - z,$$

up to an additive constant, where we used $(a + z)/(a - z) = (a + z)^2/(a^2 - z^2) = (a + z)^2/y^2$. Substituting $z = \sqrt{a^2 - y^2}$ back, we find the implicit solution

$$a \ln \left(\frac{a + \sqrt{a^2 - y(t)^2}}{|y(t)|} \right) - \sqrt{a^2 - y(t)^2} = t + C,$$

where C is an arbitrary constant.

As a check, let $G(y) = a \ln(a + z) - a \ln |y| - z$ with $z = \sqrt{a^2 - y^2}$ and $dz/dy = -y/z$. Then

$$\begin{aligned} G'(y) &= -\frac{ay}{z(a + z)} - \frac{a}{y} + \frac{y}{z} = \frac{-ay^2 - az(a + z) + y^2(a + z)}{yz(a + z)} \\ &= \frac{z(y^2 - a^2 - az)}{yz(a + z)} = \frac{-z^2(z + a)}{yz(a + z)} = -\frac{z}{y}, \end{aligned}$$

so differentiating $G(y(t)) = t + C$ with respect to t gives $-\left(\sqrt{a^2 - y(t)^2}/y(t)\right) y'(t) = 1$, which is the ODE.

Problem 7

Consider the nonlinear system of ODEs for an unknown (vector-valued) function $t \mapsto \mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))$:

$$\begin{aligned} y_1'(t) &= -0.04y_1(t) + 10^4 y_2(t)y_3(t), \\ y_2'(t) &= 0.04y_1(t) - 10^4 y_2(t)y_3(t) - 10^7 y_2(t)^2, \quad \mathbf{y}(0) = \mathbf{y}_0 \\ y_3'(t) &= 10^7 y_2(t)^2. \end{aligned}$$

- a) Show that the sum $y_1(t) + y_2(t) + y_3(t)$ of the three solution components is independent of t for $t \geq 0$. **Hint:** Do not attempt to solve this system.
- b) Prove that, for initial values $\mathbf{y}_0 \in [0, \infty)^3$, the solution $t \mapsto \mathbf{y}(t)$ remains in $[0, \infty)^3$ for every $t \geq 0$ for which the solution exists.

Solutions: a) We add the three equations. The terms on the right-hand sides cancel in pairs, so

$$(y_1(t) + y_2(t) + y_3(t))' = y_1'(t) + y_2'(t) + y_3'(t) = 0.$$

A differentiable function with zero derivative on an interval is constant. Hence, for every $t \geq 0$ for which the solution exists,

$$y_1(t) + y_2(t) + y_3(t) = y_1(0) + y_2(0) + y_3(0).$$

b) Let $\mathbf{y}_0 \in [0, \infty)^3$, and let $T > 0$ be such that the solution exists on $[0, T]$. We show that $y_1(t)$, $y_2(t)$, and $y_3(t)$ are nonnegative for every $t \in [0, T]$.

Step 1: The component y_3 . Since $y_3'(t) = 10^7 y_2(t)^2 \geq 0$, the function y_3 is nondecreasing, so $y_3(t) \geq y_3(0) \geq 0$ for every $t \in [0, T]$.

Step 2: We rewrite the equations for y_1 and y_2 . Let

$$a(t) = 10^4 y_3(t) + 10^7 y_2(t), \quad R(t) = e^{\int_0^t a(s) ds},$$

so that $R(0) = 1$, $R(t) > 0$, and $R'(t) = a(t)R(t)$. The first two equations of the system read $y_1'(t) + 0.04y_1(t) = 10^4 y_3(t)y_2(t)$ and $y_2'(t) + a(t)y_2(t) = 0.04y_1(t)$. Let

$$U(t) = e^{0.04t} y_1(t), \quad V(t) = R(t)y_2(t).$$

Then

$$\begin{aligned} U'(t) &= e^{0.04t} (y_1'(t) + 0.04y_1(t)) = 10^4 e^{0.04t} y_3(t)y_2(t) = \beta(t)V(t), \\ V'(t) &= R(t) (y_2'(t) + a(t)y_2(t)) = 0.04R(t)y_1(t) = \alpha(t)U(t), \end{aligned}$$

where

$$\alpha(t) = 0.04R(t)e^{-0.04t}, \quad \beta(t) = \frac{10^4 e^{0.04t} y_3(t)}{R(t)}.$$

The functions α and β are continuous on $[0, T]$, and they are nonnegative: $\alpha(t) > 0$, and $\beta(t) \geq 0$ by Step 1. Let $M = \max_{t \in [0, T]} \max\{\alpha(t), \beta(t)\}$. Moreover, $U(0) = y_1(0) \geq 0$ and $V(0) = y_2(0) \geq 0$.

Step 3: The functions U and V are nonnegative. For $x \in \mathbb{R}$, let $x^- = \max\{-x, 0\}$ denote the negative part of x , so that $x \geq -x^-$, and $x \geq 0$ if and only if $x^- = 0$. For $t \in [0, T]$, since $U(0) \geq 0$, $\beta \geq 0$, and $V \geq -V^-$, we have

$$U(t) = U(0) + \int_0^t \beta(s)V(s) ds \geq - \int_0^t \beta(s)V(s)^- ds \geq -M \int_0^t V(s)^- ds.$$

The right-hand side is nonpositive, so $U(t)^- \leq M \int_0^t V(s)^- ds$. In the same way, $V(t)^- \leq M \int_0^t U(s)^- ds$. Let $p(t) = U(t)^- + V(t)^-$, a continuous nonnegative function. Adding the two inequalities, we find

$$p(t) \leq M \int_0^t p(s) ds \quad \text{for every } t \in [0, T].$$

Let $P(t) = \int_0^t p(s) ds$. Then $P(0) = 0$, $P \geq 0$, and $P'(t) = p(t) \leq MP(t)$. Hence

$$(e^{-Mt}P(t))' = e^{-Mt}(P'(t) - MP(t)) \leq 0,$$

so $e^{-Mt}P(t) \leq P(0) = 0$ for every $t \in [0, T]$. Since $P \geq 0$, we conclude that $P \equiv 0$ on $[0, T]$, hence $p = P' \equiv 0$, that is, $U(t) \geq 0$ and $V(t) \geq 0$ for every $t \in [0, T]$.

Step 4: Conclusion. For every $t \in [0, T]$, we have $y_1(t) = e^{-0.04t}U(t) \geq 0$ and $y_2(t) = V(t)/R(t) \geq 0$ by Step 3, and $y_3(t) \geq 0$ by Step 1. Since T is arbitrary, the solution remains in $[0, \infty)^3$ for every $t \geq 0$ for which it exists.

Problem 8

Compute the solution to each of the following separable ODEs:

- a) $y'(t) + t \sin(t) = 0$.
- b) $t + y(t)y'(t) = 0$.
- c) $\sinh y(t)y'(t) = te^t$.
- d) $y'(t) = ty(t)(1 - y(t))$.
- e) $y'(t) - y(t)^2 = 1$.

Solutions: Throughout, C denotes an arbitrary constant.

a) The ODE reads $y'(t) = -t \sin(t)$. We integrate by parts:

$$y(t) = - \int t \sin(t) dt = t \cos(t) - \int \cos(t) dt = t \cos(t) - \sin(t) + C.$$

As a check, $y'(t) = \cos(t) - t \sin(t) - \cos(t) = -t \sin(t)$.

b) The ODE reads $y(t)y'(t) = -t$. By the chain rule, $y(t)y'(t) = (y(t)^2/2)'$, so integrating both sides gives

$$\frac{y(t)^2}{2} = -\frac{t^2}{2} + C, \quad \text{that is, } t^2 + y(t)^2 = K \quad \text{with } K = 2C.$$

A solution on an interval requires $K > 0$. Solving for y , we find

$$y(t) = \sqrt{K - t^2} \quad \text{or} \quad y(t) = -\sqrt{K - t^2}, \quad |t| < \sqrt{K}.$$

The solution curves are the upper and lower halves of the circles of radius $\sqrt{K}$ centered at the origin.

c) By the chain rule, $\sinh(y(t))y'(t) = (\cosh y(t))'$. Integrating both sides, with an integration by parts on the right, gives the implicit solution

$$\cosh y(t) = \int t e^t dt = (t - 1)e^t + C.$$

Since $\cosh$ is even and maps $[0, \infty)$ bijectively onto $[1, \infty)$ with inverse arcosh , we find

$$y(t) = \operatorname{arcosh}((t - 1)e^t + C) \quad \text{or} \quad y(t) = -\operatorname{arcosh}((t - 1)e^t + C),$$

on every interval where $(t - 1)e^t + C > 1$.

d) The constant functions $y \equiv 0$ and $y \equiv 1$ are solutions. Suppose now that $y(t) \notin \{0, 1\}$ on an interval. We separate the variables and use the partial fraction decomposition $1/(y(1 - y)) = 1/y + 1/(1 - y)$:

$$\int \frac{1}{y(1 - y)} dy = \int t dt \implies \ln \left| \frac{y(t)}{1 - y(t)} \right| = \frac{t^2}{2} + C.$$

We exponentiate. The continuous function $y/(1 - y)$ does not vanish on the interval, so it has a constant sign, and

$$\frac{y(t)}{1 - y(t)} = L e^{t^2/2}, \quad \text{where } L = e^C \text{ or } L = -e^C.$$

We solve for $y(t)$ and find $y(t) = L e^{t^2/2} / (1 + L e^{t^2/2})$. With $M = 1/L \neq 0$, this reads

$$y(t) = \frac{1}{1 + M e^{-t^2/2}}.$$

The value $M = 0$ recovers the constant solution $y \equiv 1$. If $M > -1$, the denominator is positive and the solution is defined for every $t \in \mathbb{R}$. If $M \leq -1$, the denominator vanishes at $t = \pm \sqrt{2 \ln(-M)}$, and the formula defines a solution on each interval that avoids these points. As a check, with $E(t) = M e^{-t^2/2}$, we have $E'(t) = -tE(t)$, so

$$y'(t) = -\frac{E'(t)}{(1 + E(t))^2} = \frac{tE(t)}{(1 + E(t))^2} = t \cdot \frac{1}{1 + E(t)} \cdot \frac{E(t)}{1 + E(t)} = ty(t)(1 - y(t)).$$

e) The ODE reads $y'(t) = 1 + y(t)^2$. Since $1 + y^2 > 0$, we separate the variables:

$$\int \frac{1}{1 + y^2} dy = \int 1 dt \implies \arctan y(t) = t + C.$$

Since $\arctan$ takes values in $(-\pi/2, \pi/2)$, we find

$$y(t) = \tan(t + C), \quad -\frac{\pi}{2} < t + C < \frac{\pi}{2}.$$

Problem 9

Compute the solution to each of the following linear inhomogeneous ODEs using the *variation of constants* technique:

a) $y'(t) + 2y(t) = e^{-2t} + t.$

b) $y'(t) + \frac{y(t)}{t} = \cos(t),$ where $t > 0.$

c) $y'(t) + y(t) \tan(t) = \sin(2t),$ where $t \in (-\pi/2, \pi/2).$

Solutions: In each part, we first solve the homogeneous ODE, whose solutions have the form $y(t) = LR(t)$ with a constant L . We then replace L by a function $t \mapsto L(t)$, substitute $y(t) = L(t)R(t)$ into the inhomogeneous ODE, and solve for $L(t)$. Throughout, C denotes an arbitrary constant.

a) The homogeneous ODE $y'(t) + 2y(t) = 0$ has the solutions $y(t) = Le^{-2t}$. Let $y(t) = L(t)e^{-2t}$. Then

$$y'(t) + 2y(t) = L'(t)e^{-2t} - 2L(t)e^{-2t} + 2L(t)e^{-2t} = L'(t)e^{-2t},$$

so the inhomogeneous ODE holds if and only if $L'(t)e^{-2t} = e^{-2t} + t$, that is, $L'(t) = 1 + te^{2t}$. An integration by parts gives

$$L(t) = t + \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t} + C.$$

Therefore

$$y(t) = L(t)e^{-2t} = (t + C)e^{-2t} + \frac{t}{2} - \frac{1}{4}.$$

As a check, $y'(t) = e^{-2t} - 2(t + C)e^{-2t} + 1/2$, so $y'(t) + 2y(t) = e^{-2t} + 1/2 + t - 1/2 = e^{-2t} + t$.

b) The homogeneous ODE $y'(t) = -y(t)/t$ has the solutions

$$y(t) = Le^{-\int (1/t) dt} = Le^{-\ln t} = \frac{L}{t}, \quad t > 0.$$

Let $y(t) = L(t)/t$. Then

$$y'(t) + \frac{y(t)}{t} = \frac{L'(t)}{t} - \frac{L(t)}{t^2} + \frac{L(t)}{t^2} = \frac{L'(t)}{t},$$

so the inhomogeneous ODE holds if and only if $L'(t) = t \cos(t)$. An integration by parts gives

$$L(t) = t \sin(t) + \cos(t) + C.$$

Therefore

$$y(t) = \frac{L(t)}{t} = \sin(t) + \frac{\cos(t) + C}{t}, \quad t > 0.$$

As a check, $y'(t) = \cos(t) - \sin(t)/t - (\cos(t) + C)/t^2$ and $y(t)/t = \sin(t)/t + (\cos(t) + C)/t^2$, so $y'(t) + y(t)/t = \cos(t)$.

c) On $(-\pi/2, \pi/2)$, we have $\cos(t) > 0$ and $\int \tan(t) dt = -\ln(\cos t)$. The homogeneous ODE $y'(t) = -\tan(t)y(t)$ therefore has the solutions

$$y(t) = Le^{-\int \tan(t) dt} = Le^{\ln(\cos t)} = L \cos(t).$$

Let $y(t) = L(t) \cos(t)$. Then

$$y'(t) + y(t) \tan(t) = L'(t) \cos(t) - L(t) \sin(t) + L(t) \cos(t) \tan(t) = L'(t) \cos(t),$$

so the inhomogeneous ODE holds if and only if $L'(t) \cos(t) = \sin(2t) = 2 \sin(t) \cos(t)$, that is, $L'(t) = 2 \sin(t)$. Hence

$$L(t) = -2 \cos(t) + C.$$

Therefore

$$y(t) = L(t) \cos(t) = -2 \cos^2(t) + C \cos(t), \quad t \in (-\pi/2, \pi/2).$$

As a check, $y'(t) = 4 \cos(t) \sin(t) - C \sin(t)$ and $y(t) \tan(t) = -2 \cos(t) \sin(t) + C \sin(t)$, so $y'(t) + y(t) \tan(t) = 2 \sin(t) \cos(t) = \sin(2t)$.

“Problem” 10

I would like to learn more about you. Please answer the following questions. If you don't have an answer yet for any of these, just write N/A or None.

- a) Are you an undergraduate or graduate student? At what stage are you in your studies?
What is your major or intended major?
- b) What would you like to achieve by the end of this course?
- c) What would you like to achieve by the end of your studies and career-wise (if you already know or have an idea)?
- d) Do you have any questions for me, the instructor?