

MATH 441: Solutions to Assignment 1

Problem 1

For (i)–(iii), compute the first and second derivatives dy/dx and d^2y/dx^2 . For (iv)–(vi), use implicit differentiation to compute dy/dx .

(i) $y = -2x^3 + 8x - 1$.

(ii) $y = -3x^2$.

(iii) $y = -3e^{x^2}$.

(iv) $x^2 + y^2 = 2x$.

(v) $xy^2 = 3$.

(vi) $3xy + x^2y^2 = 10$.

Solutions: (i) We differentiate term by term:

$$\frac{dy}{dx} = -6x^2 + 8, \quad \frac{d^2y}{dx^2} = -12x.$$

(ii) We have

$$\frac{dy}{dx} = -6x, \quad \frac{d^2y}{dx^2} = -6.$$

(iii) By the chain rule, $dy/dx = -3e^{x^2} \cdot 2x = -6xe^{x^2}$. By the product rule and the chain rule,

$$\frac{d^2y}{dx^2} = -6e^{x^2} - 6x \cdot 2xe^{x^2} = -6(1 + 2x^2)e^{x^2}.$$

(iv) We regard y as a function of x and differentiate both sides with respect to x :

$$2x + 2y \frac{dy}{dx} = 2.$$

At every point of the curve with $y \neq 0$, we solve for the derivative and find

$$\frac{dy}{dx} = \frac{1 - x}{y}.$$

(v) We differentiate both sides with respect to x , using the product rule on the left:

$$y^2 + 2xy \frac{dy}{dx} = 0.$$

The equation $xy^2 = 3$ forces $x \neq 0$ and $y \neq 0$, so we may divide by $2xy$ and find

$$\frac{dy}{dx} = -\frac{y}{2x}.$$

(vi) We differentiate both sides with respect to x :

$$3y + 3x \frac{dy}{dx} + 2xy^2 + 2x^2 y \frac{dy}{dx} = 0.$$

We collect the terms that contain dy/dx and factor:

$$x(3 + 2xy) \frac{dy}{dx} = -y(3 + 2xy).$$

To divide by $x(3 + 2xy)$, we check that this factor never vanishes on the curve. The product $u = xy$ satisfies $u^2 + 3u - 10 = (u + 5)(u - 2) = 0$, so $xy = 2$ or $xy = -5$. Hence $x \neq 0$ and $3 + 2xy = \pm 7 \neq 0$. Therefore

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Problem 2

- (i) Solve for x : $1 = 10 - 10e^{-x}$.
- (ii) Solve for x : $2 = 20 - 20 \ln(x)$.
- (iii) Show the equation $t = \ln |(\csc(\pi/4) + \cot(\pi/4))/(\csc x + \cot x)|$ can be inverted in terms of x to get

$$x(t) = 2 \arctan \left(\frac{e^t}{1 + \sqrt{2}} \right).$$

Hint: You will need to use trigonometric identities to prove this.

Solutions: (i) We rearrange and take logarithms:

$$1 = 10 - 10e^{-x} \iff e^{-x} = \frac{9}{10} \iff x = -\ln \left(\frac{9}{10} \right) = \ln \left(\frac{10}{9} \right).$$

(ii) For $x > 0$, we rearrange and exponentiate:

$$2 = 20 - 20 \ln(x) \iff \ln(x) = \frac{18}{20} = \frac{9}{10} \iff x = e^{9/10}.$$

(iii) Since $\csc(\pi/4) = \sqrt{2}$ and $\cot(\pi/4) = 1$, the numerator equals $1 + \sqrt{2}$. The right-hand side is defined when $\sin x \neq 0$. For such x , both $\sin(x/2) \neq 0$ and $\cos(x/2) \neq 0$, and the double-angle identities $\sin x = 2 \sin(x/2) \cos(x/2)$ and $1 + \cos x = 2 \cos^2(x/2)$ give

$$\csc x + \cot x = \frac{1 + \cos x}{\sin x} = \frac{2 \cos^2(x/2)}{2 \sin(x/2) \cos(x/2)} = \cot(x/2).$$

Hence the equation reads

$$t = \ln \left| (1 + \sqrt{2}) \tan(x/2) \right|.$$

The right-hand side is an even, 2π -periodic function of x , so we cannot invert it on its whole domain. We invert it on the interval $0 < x < \pi$, which contains $x = \pi/4$, the value that corresponds to $t = 0$. On this interval, $0 < x/2 < \pi/2$ and $\tan(x/2) > 0$, so

$$t = \ln(1 + \sqrt{2}) + \ln(\tan(x/2)).$$

The map $x \mapsto \tan(x/2)$ is a strictly increasing bijection from $(0, \pi)$ onto $(0, \infty)$, and $\ln$ is a strictly increasing bijection from $(0, \infty)$ onto $\mathbb{R}$. Therefore $x \mapsto t$ is a strictly increasing bijection from $(0, \pi)$ onto $\mathbb{R}$, and it has an inverse $t \mapsto x(t)$. To find the inverse, we exponentiate:

$$e^t = (1 + \sqrt{2}) \tan(x/2) \iff \tan(x/2) = \frac{e^t}{1 + \sqrt{2}}.$$

Since $x/2 \in (0, \pi/2)$ and $\arctan$ is the inverse of $\tan$ on $(-\pi/2, \pi/2)$, we conclude that

$$x(t) = 2 \arctan \left(\frac{e^t}{1 + \sqrt{2}} \right).$$

As a check, $t = 0$ gives $x(0) = 2 \arctan(\sqrt{2} - 1) = 2 \cdot \pi/8 = \pi/4$, because $\tan(\pi/8) = \sqrt{2} - 1$.

Problem 3

Solve the following problems involving series. You may find it helpful to review the following concepts: the direct comparison test, quotient test, Cauchy test, alternating series, and power series.

- (i) Show that $\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1$.
- (ii) Show that $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$.
- (iii) Does the series $\sum_{n=0}^{\infty} \frac{n^3}{n!}$ converge or diverge? Explain your reasoning.
- (iv) Does the series $\sum_{n=0}^{\infty} \frac{n^n}{n!}$ converge or diverge? Explain your reasoning.
- (v) Does the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converge or diverge? Explain your reasoning.
- (vi) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge or diverge? Explain your reasoning.

(vii) For what values of x does the series $\sum_{n=0}^{\infty} x^n$ converge, if any? Explain your reasoning.

Solutions: (i) Let $S_N = \sum_{j=1}^N (1/2)^j$ denote the N -th partial sum. Then

$$S_N - \frac{1}{2}S_N = \sum_{j=1}^N \left(\frac{1}{2}\right)^j - \sum_{j=2}^{N+1} \left(\frac{1}{2}\right)^j = \frac{1}{2} - \left(\frac{1}{2}\right)^{N+1},$$

so $S_N = 1 - (1/2)^N$. Since $(1/2)^N \rightarrow 0$ as $N \rightarrow \infty$, we find

$$\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = \lim_{N \rightarrow \infty} S_N = 1.$$

(ii) We first show by induction that $n! \geq 2^{n-1}$ for every $n \geq 1$. For $n = 1$, both sides equal 1. If $n \geq 2$ and $(n-1)! \geq 2^{n-2}$, then $n! = n(n-1)! \geq 2 \cdot 2^{n-2} = 2^{n-1}$. Hence $1/n! \leq (1/2)^{n-1}$ for $n \geq 1$, and, for every $N \geq 1$, part (i) gives

$$\sum_{n=0}^N \frac{1}{n!} \leq 1 + \sum_{n=1}^N \left(\frac{1}{2}\right)^{n-1} = 2 + \sum_{j=1}^{N-1} \left(\frac{1}{2}\right)^j \leq 2 + 1 = 3.$$

The terms of the series are positive, so its partial sums form an increasing sequence. This sequence is bounded above by 3, so it converges by the monotone convergence theorem. Hence $\sum_{n=0}^{\infty} 1/n! \leq 3 < \infty$.

(iii) The series converges. The term with $n = 0$ equals 0, so we consider $a_n = n^3/n!$ for $n \geq 1$. These terms are positive, and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} = \frac{(n+1)^2}{n^3} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The limit is less than 1, so the series converges by the ratio (quotient) test.

(iv) The series diverges. The term with $n = 0$ does not affect convergence, so we consider $a_n = n^n/n!$ for $n \geq 1$. These terms are positive, and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \rightarrow e \quad \text{as } n \rightarrow \infty.$$

The limit is greater than 1, so the series diverges by the ratio (quotient) test.

(v) The series converges. For $n \geq 2$, we have $\ln n > 0$, so the terms $a_n = 1/(\ln n)^n$ are positive, and

$$a_n^{1/n} = \frac{1}{\ln n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The limit is less than 1, so the series converges by the root (Cauchy) test.

(vi) The series converges. It has the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n = 1/n$. The sequence (b_n) is positive, decreasing, and tends to 0. Hence the series converges by the alternating series test (Leibniz's test).

(vii) The series converges if and only if $|x| < 1$. For $x \neq 1$, the N -th partial sum is

$$S_N = \sum_{n=0}^N x^n = \frac{1 - x^{N+1}}{1 - x},$$

which follows, as in part (i), from $S_N - xS_N = 1 - x^{N+1}$. If $|x| < 1$, then $x^{N+1} \rightarrow 0$, so $S_N \rightarrow 1/(1 - x)$ and the series converges. If $|x| \geq 1$, then $|x^n| = |x|^n \geq 1$ for every n , so the terms do not tend to 0, and the series diverges by the divergence test.

Problem 4

Compute the following integrals:

(i) $\int \frac{-\sqrt{(a^2 - y^2)}}{y} dy.$

(ii) $\int x^{3/2} dx.$

(iii) $\int ye^{-y^2} dy.$

(iv) $\int_0^{\infty} e^{-st} \cos wt dt$, where $s, w \in \mathbb{R}$.

(v) $\int_{-t/2}^{t/2} \sqrt{\frac{t^2}{4} - x^2} dx$, where $t > 0$. Hint: Use a hyperbolic trigonometric substitution to simplify the integral.

(vi) $\int \frac{a}{\sqrt{a^2 + s^2}} ds$, where $a > 0$.

(vii) $\int \frac{1}{y(K - y)} dy$, where $K > 0$.

(viii) $\int \frac{1}{1 - y^2} dy.$

Solutions: Throughout, C denotes an arbitrary constant of integration.

(i) The integrand depends on a only through a^2 , and for $a = 0$ it is undefined at every $y \neq 0$. We therefore assume $a > 0$ (for $a < 0$, replace a by $|a|$ below). The integrand is defined for

$0 < |y| \leq a$. We work on $0 < |y| < a$ and substitute $z = \sqrt{a^2 - y^2}$, so that $0 < z < a$, $z^2 = a^2 - y^2$, and $z dz = -y dy$. Then $dy = -(z/y) dz$ and

$$\int \frac{-\sqrt{a^2 - y^2}}{y} dy = \int \left(-\frac{z}{y}\right) \left(-\frac{z}{y}\right) dz = \int \frac{z^2}{y^2} dz = \int \frac{z^2}{a^2 - z^2} dz.$$

We write $z^2/(a^2 - z^2) = a^2/(a^2 - z^2) - 1$ and decompose into partial fractions:

$$\frac{a^2}{a^2 - z^2} = \frac{a^2}{(a - z)(a + z)} = \frac{a}{2} \left(\frac{1}{a - z} + \frac{1}{a + z} \right).$$

Since $0 < z < a$, both $a - z$ and $a + z$ are positive, and

$$\int \frac{z^2}{a^2 - z^2} dz = \frac{a}{2} (\ln(a + z) - \ln(a - z)) - z + C = \frac{a}{2} \ln \left(\frac{a + z}{a - z} \right) - z + C.$$

Since $(a + z)/(a - z) = (a + z)^2/(a^2 - z^2) = (a + z)^2/y^2$, the logarithm simplifies to $(a/2) \ln((a + z)/(a - z)) = a \ln((a + z)/|y|)$. Substituting $z = \sqrt{a^2 - y^2}$ back, we find

$$\int \frac{-\sqrt{a^2 - y^2}}{y} dy = a \ln \left(\frac{a + \sqrt{a^2 - y^2}}{|y|} \right) - \sqrt{a^2 - y^2} + C.$$

(ii) For $x \geq 0$, the power rule gives

$$\int x^{3/2} dx = \frac{x^{5/2}}{5/2} + C = \frac{2}{5} x^{5/2} + C.$$

(iii) We substitute $u = -y^2$, so that $du = -2y dy$:

$$\int y e^{-y^2} dy = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-y^2} + C.$$

(iv) The integral is improper: by definition, it equals $\lim_{T \rightarrow \infty} \int_0^T e^{-st} \cos(wt) dt$ whenever this limit exists. We show that the integral converges if and only if $s > 0$, in which case

$$\int_0^\infty e^{-st} \cos(wt) dt = \frac{s}{s^2 + w^2}.$$

If $s = w = 0$, then $\int_0^T 1 dt = T \rightarrow \infty$, and the integral diverges. Suppose now that $(s, w) \neq (0, 0)$, and let

$$F(t) = \frac{e^{-st} (w \sin(wt) - s \cos(wt))}{s^2 + w^2}.$$

The product rule gives

$$F'(t) = \frac{e^{-st} (-sw \sin(wt) + s^2 \cos(wt) + w^2 \cos(wt) + sw \sin(wt))}{s^2 + w^2} = e^{-st} \cos(wt),$$

so F is an antiderivative of the integrand. (One finds F by integrating by parts twice.) Since $F(0) = -s/(s^2 + w^2)$,

$$\int_0^T e^{-st} \cos(wt) dt = F(T) - F(0) = F(T) + \frac{s}{s^2 + w^2}.$$

We now let $T \rightarrow \infty$ and distinguish three cases.

Case 1: $s > 0$. Since $|F(T)| \leq e^{-sT}(|w| + |s|)/(s^2 + w^2) \rightarrow 0$ as $T \rightarrow \infty$, the integral converges to $s/(s^2 + w^2)$.

Case 2: $s = 0$. Then $w \neq 0$ and $F(T) = \sin(wT)/w$. For every positive integer k , this function takes the value 0 at $T = k\pi/|w|$ and the value $1/|w|$ at $T = (2k + 1/2)\pi/|w|$. Hence $F(T)$ has no limit as $T \rightarrow \infty$, and the integral diverges.

Case 3: $s < 0$. If $w \neq 0$, let $T_k = 2k\pi/|w|$; if $w = 0$, let $T_k = k$. In both cases, $\cos(wT_k) = 1$ and $\sin(wT_k) = 0$, so

$$F(T_k) = \frac{-se^{-sT_k}}{s^2 + w^2} \rightarrow +\infty \quad \text{as } k \rightarrow \infty,$$

because $-s > 0$. Hence $\int_0^T e^{-st} \cos(wt) dt$ has no finite limit, and the integral diverges.

(v) Let $r = t/2 > 0$, so that the integral reads $\int_{-r}^r \sqrt{r^2 - x^2} dx$. Following the hint, we substitute $x = r \tanh u$. The map $u \mapsto r \tanh u$ is a continuously differentiable, strictly increasing bijection from $\mathbb{R}$ onto $(-r, r)$, with $dx = r \operatorname{sech}^2 u du$ and, since $\operatorname{sech} u > 0$ and $1 - \tanh^2 u = \operatorname{sech}^2 u$,

$$\sqrt{r^2 - x^2} = r \sqrt{1 - \tanh^2 u} = r \operatorname{sech} u.$$

The endpoints $x = \pm r$ correspond to $u = \pm\infty$, so we first integrate over $[-b, b]$ with $0 < b < r$. With $\beta = \operatorname{artanh}(b/r)$, the substitution gives

$$\int_{-b}^b \sqrt{r^2 - x^2} dx = r^2 \int_{-\beta}^{\beta} \operatorname{sech}^3 u du.$$

To integrate $\operatorname{sech}^3 u$, we integrate by parts with $\operatorname{sech}^3 u = \operatorname{sech} u \cdot \operatorname{sech}^2 u$, using $(\tanh u)' = \operatorname{sech}^2 u$, $(\operatorname{sech} u)' = -\operatorname{sech} u \tanh u$, and $\tanh^2 u = 1 - \operatorname{sech}^2 u$:

$$\begin{aligned} \int \operatorname{sech}^3 u du &= \operatorname{sech} u \tanh u + \int \operatorname{sech} u \tanh^2 u du \\ &= \operatorname{sech} u \tanh u + \int \operatorname{sech} u du - \int \operatorname{sech}^3 u du. \end{aligned}$$

Moreover, $(\operatorname{arctan}(\sinh u))' = \cosh u / (1 + \sinh^2 u) = \cosh u / \cosh^2 u = \operatorname{sech} u$. We solve for $\int \operatorname{sech}^3 u du$ and conclude that

$$G(u) = \frac{1}{2} (\operatorname{sech} u \tanh u + \operatorname{arctan}(\sinh u))$$

is an antiderivative of $\operatorname{sech}^3 u$, and, since G is odd,

$$\int_{-b}^b \sqrt{r^2 - x^2} dx = r^2 (G(\beta) - G(-\beta)) = r^2 (\operatorname{sech} \beta \tanh \beta + \arctan(\sinh \beta)).$$

We now let $b \rightarrow r^-$. Since the integrand is continuous on $[-r, r]$, the left-hand side tends to the integral over $[-r, r]$. On the right, $\beta \rightarrow \infty$, and $\operatorname{sech} \beta \tanh \beta \rightarrow 0$ and $\arctan(\sinh \beta) \rightarrow \pi/2$ as $\beta \rightarrow \infty$. Therefore

$$\int_{-t/2}^{t/2} \sqrt{\frac{t^2}{4} - x^2} dx = \frac{\pi r^2}{2} = \frac{\pi t^2}{8}.$$

(vi) We substitute $s = a \sinh u$, a continuously differentiable bijection from $\mathbb{R}$ onto $\mathbb{R}$, with $ds = a \cosh u du$. Since $a > 0$, $\cosh u > 0$, and $1 + \sinh^2 u = \cosh^2 u$, we have $\sqrt{a^2 + s^2} = a \cosh u$ and

$$\int \frac{a}{\sqrt{a^2 + s^2}} ds = \int \frac{a}{a \cosh u} a \cosh u du = \int a du = au + C = a \operatorname{arsinh} \left(\frac{s}{a} \right) + C.$$

To express arsinh with a logarithm, let $v = \sinh u = (e^u - e^{-u})/2$. Then $e^{2u} - 2ve^u - 1 = 0$, and the positive root of this quadratic in e^u gives $e^u = v + \sqrt{v^2 + 1}$, that is, $\operatorname{arsinh} v = \ln(v + \sqrt{v^2 + 1})$. With $v = s/a$,

$$a \operatorname{arsinh} \left(\frac{s}{a} \right) = a \ln \left(\frac{s + \sqrt{a^2 + s^2}}{a} \right) = a \ln \left(s + \sqrt{a^2 + s^2} \right) - a \ln a.$$

We absorb $-a \ln a$ into the constant and find

$$\int \frac{a}{\sqrt{a^2 + s^2}} ds = a \ln \left(s + \sqrt{a^2 + s^2} \right) + C.$$

(vii) The integrand is defined for $y \notin \{0, K\}$. We decompose into partial fractions:

$$\frac{1}{y(K-y)} = \frac{1}{K} \left(\frac{1}{y} + \frac{1}{K-y} \right),$$

which one checks by putting the right-hand side over the common denominator $y(K-y)$. Hence

$$\int \frac{1}{y(K-y)} dy = \frac{1}{K} (\ln |y| - \ln |K-y|) + C = \frac{1}{K} \ln \left| \frac{y}{K-y} \right| + C.$$

(viii) The integrand is defined for $y \notin \{-1, 1\}$. We decompose into partial fractions:

$$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{1}{2} \left(\frac{1}{1-y} + \frac{1}{1+y} \right).$$

Hence

$$\int \frac{1}{1-y^2} dy = \frac{1}{2} (-\ln |1-y| + \ln |1+y|) + C = \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| + C.$$