# MATH 441: Homework 6

This homework is due on Friday May 1, in-class.

Problems 4, 5, 6, and 7 are due to Strogatz (2001).

## Readings

Boyce and diPrima: Sections 2.7-2.8, 3.7-3.8, 7.1-7.8, 8.1, 8.6 and 9.1-9.5.

## Problem 1

Consider the initial value problem

$$y'(t) = \sqrt{y(t)}, \quad y(0) = y_0.$$

a) Show that if $y_0 = 0$, the problem above has more than one solution, and find these solutions.

b) Show that if $y_0 > 0$, the problem above has a unique solution, and find this solution.

c) Briefly discuss your results in the context of Picard–Lindelöf's Theorem.

## Problem 2

Consider the initial value problem

$$y'(t) = y(t)^2, \quad y(0) = 1.$$

Show the solution to this problem exists and is unique, but blows up in "finite time", in the sense that the solution exists only on some interval $t \in [0, t_1)$. Determine $t_1$, and investigate the limit $\lim_{\substack{t \to t_1 \\ t < t_1}} y(t)$.

## Problem 3

Apply the Picard–Lindelöf iterations to the problem

$$y'(t) = ay(t), \quad y(0) = 1.$$

In addition, show, without using Picard–Linderlöf's Theorem, that the resulting sequence of solutions $\{y_n\}_{n=0}^{\infty}$ converges to the exact solution of the problem.

## Problem 4 (Rabbits vs. Sheep)

In this problem you will study a simple form of the classic **Lotka–volterra model of competition** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. We will ignore all other complications (predators, seasonal effects, other sources of food, etc.) There are two main effects to consider:

1. Each species grows to its carrying capacity in the absence of the other. Rabbits reproduce faster, so they should have an intrinsic faster growth rate than sheep.
2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling on the grass. We'll assume that these conflicts occur at a rate proportional to the size of each population. Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

A specific model that incorporates these assumptions is

$$R'(t) = R(3 - R - 2S)$$
$$S'(t) = S(2 - R - S)$$

and $R, S \geqslant 0$.

a) Find the fixed points for the system.

b) Compute the Jacobian of each fixed point.

c) Classify the stability of each fixed point.

d) Using c), draw the phase portrait for the system.

e) Interpret the phase portrait. Can the rabbits and sheep coexist? Why or why not? if not, what determines which species survive?

## Problem 5 (The pendulum)

In the absence of damping and external driving, the motion of a pendulum is governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0 \tag{1}$$

where $\theta$ is the angle from the downward vertical, $g$ is the acceleration due to gravity, and $L$ is the length of the pendulum. We now non-dimensionalize (1) by introducing a frequency $\omega = \sqrt{g/L}$ and a dimensionless time $\tau = \omega t$. Then the equation becomes

$$\dot{\theta}(\tau) + \sin(\theta) = 0$$

where the overdot denotes differentiation with respect to $\tau$. The corresponding system in the phase plane is

$$\dot{\theta} = \nu$$
$$\dot{\nu} = -\sin(\theta)$$

where $\nu$ is the (dimensionless) angular velocity.

a) Show that the fixed points of this system are given by $(\theta^*, \nu^*) = (k\pi, 0)$, where $k$ is any integer. Since there is no physical difference between angles that differ by $2\pi$, we'll concentrate on the two fixed points $(0, 0)$ and $(\pi, 0)$.

b) Show that the origin $(0, 0)$ is a linear center. (In fact, it is a nonlinear center – the proof is beyond the scope of this course, but roughly it relies on the fact that the pendulum conserves energy.)

c) Show that the fixed point $(\pi, 0)$ is a saddle. Compute the corresponding eigenvectors.

d) Using c) and d), draw the phase portrait **near the fixed points**.

e) (Optional) Draw the whole phase portrait. Here, note that if $\nu$ is large enough (that is, we swing the pendulum hard and give it lots of energy) the pendulum will whirl repeatedly over the top (because there is no friction in this problem).

## Problem 6: Hamiltonian Systems I

Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newton's laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest.

Here's the simplest instance of a Hamiltonian system. Let $H(p, q)$ be a smooth, real-valued function of two variables. The variable $q$ is the "generalized coordinate" and $p$ is the "conjugate momentum." (In some physical settings, $H$ could also depend explicitly on time $t$, but we'll ignore that possibility.) Then a system of the form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

is called a **Hamiltonian system** and the function $H$ is called the **Hamiltonian**. These equations for $\dot{q}$ and $\dot{p}$ are called Hamilton's equations.

a) Show that for any Hamiltonian system, the quantity $H(q, p)$ is a conserved quantity.

b) (Harmonic oscillator) For a simple harmonic oscillator of mass $m$, spring constant $k$, displacement $q$, and momentum $p$, the Hamiltonian is

$$H(q, p) = \frac{p^2}{2m} + \frac{kq^2}{2}.$$

Write out Hamilton's equations explicitly. Show that one equation gives the usual definition of momentum and the other is equivalent to $F = ma$.

c) Find the equilibrium solution to Hamilton's equations and classify its type.

d) Draw the phase portrait.

## Problem 7: Hamiltonian Systems II

A particle moves in a plane under the influence of an inverse-square force. It is governed by the Hamiltonian

$$H(r,p) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r},$$

where $r > 0$ is the distance from the origin and $p$ is the radial momentum. The parameters $h$ and $k$ are the angular momentum and force constant, respectively.

a) Suppose $k > 0$, corresponding to an attractive force like gravity. Sketch the phase portrait in the $(r,p)$ plane. **Hint:** Graph the "effective potential" $V(r) = h^2/2r^2 - k/r$ and then look for intersections with horizontal lines of height $E$. Use this information to sketch the contour curves $H(r,p) = E$ for various positive and negative values of $E$.

b) Show that the trajectories are closed if $-k^2/2h^2 < E < 0$, in which case the particle is "captured" by the force. What happens if $E > 0$? What about $E = 0$?

c) If $k < 0$ (as in electric repulsion), show that there are no periodic orbits.