MATH 441: Homework 5

This homework is due on Wednesday April 15, in-class.

Readings

Boyce and diPrima: Sections 7.1-7.8.

Problem 1

Consider the system $x'(t) = x - y, y'(t) = x + y$.

a) Write the system as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Show that the roots of the associated characteristic polynomial (eigenvalues of $\boldsymbol{A}$) are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, with eigenvectors $v_1 = (i, 1)$, $v_2 = (-i, 1)$. **Note:** The eigenvalues are complex conjugates, and so are the eigenvectors – this is always the case for real $\boldsymbol{A}$ with complex eigenvalues.

b) Letting $\boldsymbol{x}(t) = (x(t), y(t))$, the general solution is $\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2$. So in one sense, we're done. But this way of writing $\boldsymbol{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\boldsymbol{x}(t)$ purely in terms of real-values functions. **Hint:** Use the identity

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

to rewrite $\boldsymbol{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.

Problem 2

Consider the matrix

$$\boldsymbol{A} = \frac{1}{4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

a) Show that $\boldsymbol{A}$ is not diagonalizable.

b) Calculate the matrix exponential $e^{\boldsymbol{A}t}$. **Hint:** Compute $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, $\boldsymbol{A}^4$, $\boldsymbol{A}^5$, and use your results to find a general formula for $\boldsymbol{A}^k$. Then use the matrix exponential definition

$$e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!}$$

to calculate $e^{\boldsymbol{A}t}$.

c) Find the general solution of the system $\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t)$.

Problem 3

Compute the resolvent matrices $\boldsymbol{R}(t, t_0)$ for each of the two systems (with $\boldsymbol{y}(t_0) = \boldsymbol{y}_0 \in \mathbb{R}^2$):

$$\text{(i)} \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ -2y_2(t) \end{pmatrix}, \quad (ii) \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ -y_1(t). \end{pmatrix}.$$

Problem 4

Find the general solution of

$$t \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

using diagonalization. **Hint:** Use the substitution $t = e^s$, $y_1(t) = z_1(s)$ and $y_2(t) = z_2(s)$.

Problem 5

Solve the set of coupled second-order ODEs

$$\begin{aligned} y_1''(t) &= -6y_1(t) + 4y_2(t) \qquad & y_1(0) = 3, \quad y_2(0) = 0 \\ y_2''(t) &= 2y_1(t) - 4y_2(t) \qquad & y_1'(0) = 0, \quad y_2'(0) = 0 \end{aligned}$$

a) Using the Laplace transform.

b) By transforming it into a first-order system of ODEs.

Problem 6

Consider the differential equation for a certain electric circuit,

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = 0,$$

where $t \mapsto I(t)$ is the current (at time t), $R > 0$ the resistance, $C > 0$ the capacitance, and $L > 0$ the inductance.

a) Write the ODE above as a first-order system

b) Analyze the nature and stability of the critical point $(0, 0)$ as a function of the constants L, C, and R.

Problem 7

Consider the system $x'(t) = 4x - y, y'(t) = 2x + y$.

a) Write the system as
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$
Find the characteristic polynomial of $\boldsymbol{A}$ and compute the corresponding eigenvalues and eigenvectors.

b) Find the general solution of the system.

c) Classify the stability type of the equilibrium solution
$$\begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

d) Solve the system subject to the initial condition $(x(0), y(0)) = (3, 4)$ and draw this solution on the $(x(t), y(t))$-plane.