MATH 441: Homework 2

Due **in-class** on Wednesday February 25, 1:00 PM.

Readings

Boyce and diPrima: Sections 2.4-2.6 and 5.1-5.6.

Problem 1

Compute the solution to each of the following separable ODEs:

a) $y'(t) = e^{-2y(t)}/y(t)$.

b) $y'(t) = y(t)(1 + y(t))$.

c) $y'(t) = \sec(y(t)^3)t^2/3y(t)^2$

Problem 2

A function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called *homogeneous of degree one* if

$$f(\lambda t, \lambda y) = f(t, y) \text{ for every } \lambda \in \mathbb{R}.$$

a) Consider the ODE

$$y'(t) = f(t, y(t)),$$

where f is function that is homogeneous of degree one. Show that with the substitution $u(t) = y(t)/t$ there holds (informally)

$$\int \frac{1}{f(1, u) - u}\, du = \ln(t) + C,$$

where C is an arbitrary constant.

b) Use a) to solve the ODE

$$y'(t) = \frac{y(t)^2}{ty(y) + t^2}.$$

Leave your solution in implicit form.

Problem 3

(a) Solve the logistic equation ODE:

$$N'(t) = rN(t)(1 - N(t)/K), \; N(0) = N_0,$$

where $r > 0$, $K > 0$, and $N_0 > 0$. **Hint:** To solve the ODE, you can either solve it directly using partial fractions or you can make the change of variable $x(t) = 1/N(t)$ and then derive and solve the corresponding ODE for $x(t)$.

b) Find the limit $\lim_{t\to+\infty} N(t)$ analytically.

Problem 4

Consider the ODE

$$4ty''(t) + 2y'(t) + y(t) = 0.$$

a) Find the (regular) singular point t_0 of the given equation, state the indicial equation and find its roots.

b) Find a fundamental set of Frobenius series solution for $t > t_0$ expanded about the singular point t_0. Your answer should include an expression for the general n^{th} coefficient in each series solution, not just a recursion relation.

c) For what value(s) of α is it possible to satisfy the initial value problem $y(t_0) = 1$, $y'(t_0) = \alpha$ for this equation?

Problem 5

Find a series solution (with two free parameters) of the following ODE problem close to the point $t_0 = 0$:

$$(1 - t)y''(t) + ty'(t) - y(t) = 0.$$

In addition, find an explicit formula for the coefficients and compute the convergence radius of the resulting series.

Problem 6

Compute the general solution to each of the following ODEs:

a) $t^2y''(t) - 2ty'(t) + y(t) = 0.$

b) $t^2y''(t) - 3ty'(t) + 4y(t) = 0.$

c) $t^2y''(t) - 4ty'(t) + 6y(t) = 0.$