MATH 441: Homework 1

This homework is due on Wednesday February 11, 5 PM.

Readings

The readings are taken from *Elementary Differential Equations, 12th edition, by W.E. Boyce and R.C. DiPrima.* Previous editions are just as good; just find the corresponding pages and materials.

Boyce and diPrima: Sections 1 and 2.1-2.3

Problem 1

Consider the Bernoulli equation

$$y'(t) + f(t)y(t) = g(t)y(t)^n.$$

Using the coordinate transformation $\varphi(t) = y(t)^q$ and a suitable choice of q, turn the Bernoulli equation into a linear equation. (Do not solve the resulting ODE.)

Problem 2

For which $\lambda > 0$ does the following ODE have a unique solution?

$$y'(t) - y(t)^\lambda = 0, \ y(0) = 0, \text{ with } t \geqslant 0?$$

Problem 3

Consider the initial value problem (due to Augustin-Louis Cauchy, 1824):

$$(y'(t))^3 = 8y(t)^2 - 4ty(t)y'(t), \ y(0) = y_0, \tag{1}$$

where $y_0 \geqslant 0$ is a given initial value. Show there exists a constant $C \in \mathbb{R}$ such that $y(t) = C(t + C)^2$ is a solution to the problem above. For $y_0 = 0$, find yet another solution of (1), that is, show the solution is not unique.

Problem 4

Consider the initial value problem

$$ty'(t) + \lambda y(t) = g(t), \; y(0) = y_0,$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and λ, $y_0 \in \mathbb{R}$. Show that the solution to this problem, provided it exists, is unique for $t \geqslant 0$. **Hint:** Suppose there are two solutions y_1, y_2 and define $w = y_1 - y_2$. Prove that $w \equiv 0$, that is, $w(t) = 0$ identically on $t \geqslant 0$.

Note: The hint given in this problem is a standard technique for proving (or disproving) uniqueness of solutions to ODEs (and, in fact, other types of differential equations). We will use this technique again in the course, and those taking MATH 442 and further mathematics courses will encounter it again.

Problem 5

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, with at least one zero in $\mathbb{R}$, and let $y_0 \in \mathbb{R}$. Consider the implicit ODE problem

$$y'(t)f'(y(t)) + f(y(t)) = 0, \; y(0) = y_0.$$

Assume there exists a sufficiently smooth solution $t \mapsto y(t)$ to this problem above, with $f(y(t)) \neq 0$ over $t \in [0, \infty)$. Prove that $y(t)$ converges to a zero of f as $t \to \infty$, that is,

$$f(\lim_{t\to\infty} y(t)) = 0.$$

Hint: You may find it helpful to use the chain rule: $(f(y(t)))' = y'(t)f'(y(t))$.

Problem 6

A pond initially contains $1,000,000$ gallons of water and an unknown amount of undesirable chemical. Water containing 0.01gram/gallon flows into the pond at a rate of 300 gallons/hour. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

a) Write a differential equation for the amount of chemical $c(t)$ as a function of t.

b) Solve the differential equation for $c(t)$.

c) What happens as $t \to \infty$? Does the initial amount of the chemical compound in the pond matter?

Problem 7

Find the implicit solution of the equation (due to Gottfried Wilhelm Liebniz):

$$y'(t) = -\frac{y(t)}{\sqrt{a^2 - y(t)^2}}, \; a > 0.$$

Hint: Use the substitution $z^2 = a^2 - y^2$, $z\,dz = -y\,dy$, in the integration.

Problem 8

Consider the nonlinear system of ODEs for an unknown (vector-valued) function $t \mapsto \boldsymbol{y}(t) = (y_1(t), y_2(t), y_3(t))$:

$$\begin{aligned} y_1'(t) &= -0.04y_1(t) + 10^4 y_2(t)y_3(t), \\ y_2'(t) &= 0.04y_1(t) - 10^4 y_2(t)y_3(t) - 10^7 y_2(t)^2, \;\; \boldsymbol{y}(0) = y_0 \\ y_3'(t) &= 10^7 y_2(t)^2. \end{aligned}$$

a) Show that the sum $y_1(t) + y_2(t) + y_3(t)$ of the three solution components is constant for every $t \geqslant 0$. **Hint:** Do not attempt to solve this system.

b) Prove that, for initial values $\boldsymbol{y}_0 \in [0, +\infty)^3$, the solution $t \mapsto \boldsymbol{y}(t)$ remains in $[0, \infty)^3$ for every $t \geqslant 0$ for which the solution exists.

Problem 9

Compute the solution to each of the following separable ODEs:

a) $y'(t) + t\sin(t) = 0$.

b) $t + y(t)y'(t) = 0$.

c) $\sinh y(t)y'(t) = te^t$.

d) $y'(t) = ty(t)(1 - y(t))$.

e) $y'(t) - y(t)^2 = 1$.

"Problem" 10

I would like to learn more about yourself. Please answer the following questions. If you don't have an answer yet for any of these, just write N/A or None.

a) Are you an undergraduate or graduate student? At what stage are you in your studies? What is your major or intended major?

b) What would you like to achieve by the end of this course?

c) What would you like to achieve by the end of your studies and career-wise (if you already know or have an idea)?

d) Do you have any questions for me, the instructor?

Note: Ordinary differential equations is a vast and important subject that finds applications across mathematics, science and engineering. This includes traditional fields such as geometry, physics, economics, biology and engineering, and more "trendy" fields such as machine learning, deep learning, and data science.

I intend to cover applications across these fields, but depending on your interests I can adjust accordingly and design applied problems in your field(s) of interest.