MATH 441: Homework 0

This problem set, which will not be graded and which you do not have to turn in, is meant to give you an idea of the material from differential and integral calculus you should be comfortable with in order to do well in MATH 441.

Problem 1

For (i)-(iii), compute dy/dx and d^2y/dx^2. For (iv)-(vi), use implicit differentiation to compute dy/dx.

(i) $y = -2x^3 + 8x - 1$.

(ii) $y = -3x^2$.

(iii) $y = -3e^{x^2}$.

(iv) $x^2 + y^2 = 2x$.

(v) $xy^2 = 3$.

(vi) $3xy + x^2y^2 = 10$.

Problem 2

(i) Solve for x: $1 = 10 - 10e^{-x}$.

(ii) Solve for x: $2 = 20 - 20\ln(x)$.

(iii) Show the equation $t = -\ln|(\csc(\pi/4) + \cot(\pi/4))/(\csc x + \cot x)|$ can be inverted in terms of x to get

$$x(t) = 2\arctan\left(\frac{e^t}{1+\sqrt{2}}\right).$$

Hint: You will need to use trigonometric identities to prove this.

Problem 3

Solve the following problems involving series. You may find it helpful to review the following concepts: the direct comparison Test, quotient test, Cauchy test, alternating series, and power series.

(i) Show that $\sum_{j=1}^{n} \left(\frac{1}{2}\right)^j = 1$.

(ii) Show that $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$.

(iii) Does the series $\sum_{n=0}^{\infty} \frac{n^3}{n!}$ converge or diverge? Explain your reasoning.

(iv) Does the series $\sum_{n=0}^{\infty} \frac{n^n}{n!}$ converge or diverge? Explain your reasoning.

(v) Does the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converge or diverge? Explain your reasoning.

(vi) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge or diverge? Explain your reasoning.

(vii) For what values of x does the series $\sum_{n=0}^{\infty} x^n$ converge, if any? Explain your reasoning.

Problem 4

Compute the following integrals:

(i) $\displaystyle\int \frac{-\sqrt{(a^2 - y^2)}}{y}\, dy.$

(ii) $\displaystyle\int x^{3/2}\, dx.$

(iii) $\displaystyle\int ye^{-y^2}\, dy.$

(iv) $\displaystyle\int_0^{\infty} e^{-st} \cos wt\, dt$, where $s,\ w \in \mathbb{R}$.

(v) $\displaystyle\int_{-t/2}^{t/2} \sqrt{\frac{t^2}{4} - x^2}\, dx$, where $t > 0$. Hint: Use a hyperbolic trigonometric substitution to simplify the integral.

(iv) $\displaystyle\int \frac{a}{\sqrt{a^2 + s^2}}\, ds$, where $a > 0$.

(vii) $\displaystyle\int \frac{1}{y(K - y)}\, dy$, where $K > 0$.

(viii) $\displaystyle\int \frac{1}{1 - y^2}\, dy.$

Problem 5

Verify that the function is a solution to the differential equation.

	Solution	Differential equation
(i)	$y = 3x^2$	$y' = 3x^2$
(ii)	$y = e^{-2x}$	$y' + 2y = 0$
(iii)	$y = x^2$	$x^2y'' - 2y = 0$
(iv)	$y = 1/x$	$xy'' + 2y' = 0$

Verify that the function is a solution to the differential equation for any value of C.

(v)	$y = Ce^{4x}$	$y' = 4x$
(vi)	$y = Cx^2 = 3x$	$xy' - 3x - 2y = 0$
(vii)	$y = x \ln x^2 + 2x^{3/2} + Cx$	$y' - \frac{y}{x} = 2 + \sqrt{x}$

Verify that the function is a solution to the differential equation using implicit differentiation for any value of C.

(viii)	$x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
(iv)	$x^2 - y^2 = C$	$y^3y'' + x^2 - y^2 = 0.$

Problem 6

In Problem 4, you may have computed some of the integrals using the change of variables technique, also sometimes called integration by substitution. This technique is justified by the following Theorem:

Theorem. *Let $g : [a, b] \to I$ be a differentiable function with a continuous derivative, where $I \subset \mathbb{R}$ is an interval. Suppose that $f : I \to \mathbb{R}$ is a continuous function. Then:*

$$\int_a^b f(g(x)) \frac{dg}{dx}(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

This theorem is also useful for solving so-called separable ordinary differential equations (ODEs) of the form

$$\frac{dy}{dx}(x) = f_1(x) f_2(y(x)).$$

If $f_2(y(x)) \neq 0$, we can rearrange to obtain

$$\frac{1}{f_2(y(x))} \frac{dy}{dx}(x) = f_1(x).$$

This is often written as $1/f_2(y)dy/dx = f_1(x)$, which makes it more clear why the ODE is called separable. Now, assume that $x \mapsto y(x)$ is differentiable with a continuous derivative over some interval $[x_0, x_1]$, and assume that f_2 is a continuous function[1]. We can integrate

$$\int_{x_0}^{x_1} \frac{1}{f_2(y(x))} \frac{dy}{dx}(x)\, dx = \int_{x_0}^{x_1} f_1(x)\, dx$$

and apply the theorem above to the integral on the left (with $f \equiv 1/f_2$ and $g \equiv y$) to find

$$\int_{x_0}^{x_1} \frac{1}{f_2(y(x))} \frac{dy}{dx}(x)\, dx = \int_{y(x_0)}^{y(x_1)} \frac{1}{f_2(y)}\, dy.$$

In other words, we can "cancel out" the dx terms in the numerator and denominator. Hence, we find the implicit representation

$$\int_{y(x_0)}^{y(x_1)} \frac{1}{f_2(y)}\, dy = \int_{x_0}^{x_1} f_1(x).$$

Formally, the same holds for indefinite integration:

$$\int \frac{1}{f_2(y)}\, dy = \int f_1(x).$$

If we can integrate the two integrals, then we can get an explicit (or implicit) solution of $y(x)$ in terms of x. (Don't forget the integration constant!)

Exercises. Compute the solution to each of the following separable ODEs:

(i) $y'(t) + t\sin(t) = 0$.

(ii) $t + y(t)y'(t) = 0$.

(iii) $\sinh y(t)y'(t) = te^t$.

(iv) $ty'(t) + y(t) = 0$.

(v) $y'(t) - y(t)^2 = 1$.

[1] We will see later when we can guarantee these conditions via *existence* results for ODEs.