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Lecture Notes : APMA 0350 Applied Ordinary Differential Equations

O → in one variable

D → that involves some of the derivatives of an unknown function.

E → an ODE is an equation of an unknown function and its derivatives.

Applications : ODEs are used to model and describes processes
- Physics and engineering (mass-spring, aircrafts, climate prediction)
- Chemistry (reaction networks and chemical dynamics)
- Biology (epidemic models, cancer/tumor growth, ecology)
- Economics

$\left\{\begin{array}{l} \text{model (DE form)} \\ \text{ODE} \\ \text{solutions} \end{array}\right\}$ => (Application → model → ODE → solutions)

Solutions :
- Analytical solutions
- Solution properties
- Numerical solutions

We will focus on the analytical aspects of ODEs in this course. We will also discuss applications of ODEs and briefly touch numerical solutions and solution properties.

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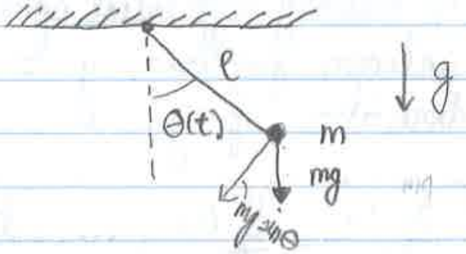
§ 1. Introduction

§ 1.1. Terminology and basics

Application → model → ODE → Solutions

Ex : Pendulum



t : time
$\theta(t)$: Angle at time t
m : mass of the pendulum
ℓ : Length of the pendulum
g : Gravity constant

Model : Newton's laws of motion

$$\Rightarrow F(t) = ma(t) \text{ with } a(t) = -\frac{d^2\theta}{dt^2}$$

Torque : $$-\frac{d^2\theta(t)}{dt^2} m\ell = mg\sin(\theta(t))$$

$$\Rightarrow \frac{d^2\theta(t)}{dt^2} + \frac{g}{\ell}\sin(\theta(t)) = 0.$$

Let $\omega^2 = g/\ell$ (>0) ↖ frequency

$$\Rightarrow \boxed{\frac{d^2\theta(t)}{dt^2} + \omega^2\sin(\theta(t)) = 0}$$

This is a nonlinear equation, and it is unsolvable! However,

$$\sin(\theta(t)) \approx \theta(t) \quad \text{for small angles.}$$

(Recall the Maclaurin expansion of $\sin x$: $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

$$\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$)

This leads to the linearized ODE
(Harmonic oscillator)

$$\boxed{\frac{d^2\theta(t)}{dt^2} + \omega^2\Theta(t) = 0}$$

The general solution is $\Theta(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$, where C_1 and C_2 are (a priori undetermined) numbers.

Defn: An ordinary differential equation (ODE) is an equation for an unknown function $y : I \geq \mathbb{R} \to \mathbb{R}$ that involves some derivatives of y:

$$\Phi\left(\frac{d^ny}{dt^n}(t), \frac{d^{n-1}y}{dt^{n-1}}(t), \dots, \frac{d^2y}{dt^2}(t), \frac{dy}{dt}(t), y(t), t\right) = 0$$

for some function Φ.

- $n \geq 1$ is called the order of the ODE.
- y is called the dependent variable (it depends on t).
- t is called the independent variable.

An ODE is called linear if it has the form

$$a_n(t)\frac{d^ny}{dt^n}(t) + \dots + a_1(t)\frac{dy}{dt}(t) + a_0(t)y(t) = f(t).$$

Otherwise, the ODE is nonlinear.

Remark: An n^{th}-order ODE can be transformed into a system of 1^{st}-order ODEs of n equations.

Break =

Ex: Consider $\frac{d^2\theta}{dt^2}(t) + \omega^2\theta(t) = 0$.

$$\text{Let } \left.\begin{aligned} y_1(t) &= \Theta(t) \\ y_2(t) &= \frac{d\Theta}{dt}(t) \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} \frac{dy_1}{dt}(t) &= y_2(t) \\ \frac{dy_2}{dt}(t) &= -\omega^2 y_1(t) \end{aligned}\right.$$

This can be written in matrix form as

$$\begin{pmatrix} \frac{dy_1(t)}{dt} \\ \frac{dy_2(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

Examples

a) $\frac{dy}{dx}(x) = 1-3x+y(x)+x^2+x\,y(x)$. This is a 1st order ODE (Newton, 1671)

b) $\frac{dy}{dt}(t) = -y(t) \Big/ \sqrt{a^2-(y(t))^2}$. This is a 1st order nonlinear ODE (Leibniz, Bernoulli 1690).

(You will solve this ODE in assignment 1 ...)

c) $y(t) - t\frac{dy}{dt}(t) + f(y'(t)) = 0$ ((Clairaut, 1734).

This is a 1st order, in general nonlinear implicit ODE.
→ (Highest order derivative appears implicitly.)

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Let's look at the following ODE:

$$\frac{dy}{dt}(t) = 2t \quad \Longrightarrow \quad y(t) = t^2 + C$$ for any number C.

⟹ This ODE has infinitely many solutions!
⟹ We need additional conditions to make this solution unique!

e.g.: $\frac{dy}{dt}(t) = 2t$ and $y(0) = 4$.

$$\Rightarrow y(t) = t^2 + 4.$$

⟹ $\boxed{\text{Typically, a } n^{th}\text{-order ODE requires } n \text{ additional conditions.}}$

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In practical applications:

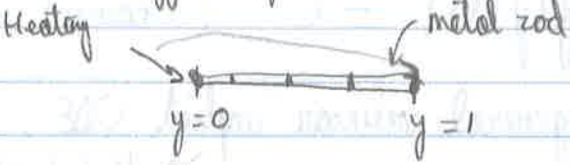
a) For an n^{th}-order ODE, prescribe the solution and its first n-1 derivatives at one point (evolution process).

E.g.: $\frac{d^3y}{dt^3}(t) + t\frac{d^2y}{dt^2}(t) + 9\frac{dy}{dt} + 9ty = 0$ with $y(0)=2$, $\frac{dy}{dt}(0) = 0$, $\frac{d^2y}{dt^2}(0) = -18$.

$\Rightarrow$ Unique solution is $y(t) = 2\cos(3t)$.

This is called an initial value problem (IVP).

b) Prescribe the solution at different points (for a 2^{nd}-order ODE, at two different points).



E.g.: $\frac{d^2y}{dt^2}(t) + \frac{dy}{dt}(t) = \cos(t)$ with $y(0) = y(1) = 0$.

$\Rightarrow$ Unique solution is $y(t) = \frac{1}{2}(t-1)\sin t$

This is called a boundary value problem (BVP).

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A word on notation: There is no single uniform notation for differentiation...

Leibniz's notation : $dy/dt\,(t)$

Lagrange's notation : $y'(t)$

Newton's notation : $\dot{y}(t)$ (Newton's notation is generally used when the independent variable denotes time.)

(From now on, we'll mostly use Lagrange's notation.)
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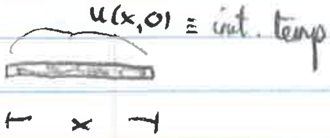
Remark: There are other types of differential equations. For example:

a) Partial differential equations (PDEs). This is a DE for a function with more than one independent variable. For example, the Heat equation

Let $u(x,t)$ denote the temperature of a rod at position x and time t.

1D Heat PDE: $\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2}{\partial x^2} u(x,t) = 0$



b) Integro-differential equations (IDEs). For example, the linear Volterra equation of the second kind

$$y'(t) + \int_0^t K(t,s)\, y(s)\, ds = f(t),$$ where K and f are given.

(e.g., this is used in polymer physics, ecology, water waves, quantum mechanics...)

c) Delay differential equations (DDEs). This is a DE for a function in which the derivative at a time t depends on the value of a function at a previous time before time t. For example, the Mackey-Glass equation

$$\frac{dx}{dt}(t) = \frac{\beta x}{1+x(t-\tau)} - \gamma x,$$ β, γ, and τ are real numbers.

(This was originally used by Michael C. Mackey and Leon Glass in 1977 to model and illustrate the appearance of complex dynamics in the physiology of mammals!)

Summary :

1) ODE ≡ equation for an unknown function in one variable that involves the derivative(s) of that function.

2) A linear ODE has the form :

$$a_n(t)\, y^{(n)}(t) + \cdots + a_2(t)\, y^{(2)}(t) + a_1(t)\, y^{(1)}(t) + a_0(t)\, y(t) = \beta(t).$$

Otherwise, it is called nonlinear.

For the linear case : $\cdot\ \beta \equiv 0 \leftrightarrow$ ODE is homogeneous

$\cdot\ \beta \not\equiv 0 \leftrightarrow$ ODE is nonhomogeneous

3) Typically, ODE solutions are not unique.

↳ Impose additional conditions (IVP, BVP)

§1.2. Basic tools for solving ODEs

§1.2.1 First order ODEs

We will consider two types of first order ODEs

① $y'(t) = f(t)\,g(y(t))$, where f and g are given functions

② $P(t, y(t)) + Q(t, y(t))\,y'(t) = 0$, where P and Q are given functions.

We call ① a separable ODE and ② a total ODE.

① $y'(t) = f(t)\,g(y(t))$, where f and g are given functions.

Note that we can write this as $\frac{1}{g(y(t))} y'(t) = f(t)$ whenever $g(y(t)) \neq 0$.

Idea: Express both sides of the ODEs in terms of derivatives in t
We may do so via the fundamental theorem of calculus:

$$f(t) = \frac{d}{dt} \int_{t_0}^{t} f(\vartheta)\, d\vartheta$$

and

$$\frac{1}{g(t)} y'(t) = \frac{d}{dt} \int_{y(t_0)}^{y(t)} \frac{1}{g(z)}\, dz$$

This leads to $\frac{d}{dt}\left(\int_{y(t_0)}^{y(t)} \frac{1}{g(z)}\,dz\right) = \frac{d}{dt}\left(\int_{t_0}^{t} f(\tau)\,d\tau\right)$

$$\Rightarrow \underbrace{\int_{y_0}^{y(t)} \frac{1}{g(z)}\,dz}_{\substack{\text{Gives } G(y(t)) \text{ (with } \frac{d}{dt}G(y(t)) = \frac{1}{g(t)}y'(t)\text{), and} \\ \text{we hope we can solve for } y(t)\ldots}} = \int_{t_0}^{t} f(\tau)\,d\tau \;+\; \overset{\nwarrow \text{ constant}}{C}$$

This technique is known as separation of variables.

Ex: (Leibniz equation) $y'(t) = -\underbrace{\frac{y(t)}{\sqrt{a^2 - y(t)^2}}}_{g(y(t))} \cdot \underbrace{1}_{f(t)}$

$$\Rightarrow -\int_{y(t_0)}^{y(t)} \frac{\sqrt{a^2 - z^2}}{z}\,dz = (t - t_0) + C$$

(HW): You will solve this! Hint: (Use the substitution $v^2 = a^2 - z^2$, $v\,dv = -z\,dz$ in the integral!)

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Special cases:

i) $y'(t) = f(t)\,y(t)$. Here, $g(z) = z$, and so we get

$$\frac{1}{y(t)}\,y'(t) = f(t), \text{ leading to}$$

$$\underbrace{\int_{y(t_0)}^{y(t)} \frac{1}{z}\,dz}_{=\ln(y(t)/y_0)} = \int_{t_0}^{t} f(\tau)\,d\tau + C$$

$$\Rightarrow \ln(y(t)) = \int_{t_0}^{t} f(\tau)\,d\tau + (C + \ln(y_0))$$

$$\Rightarrow y(t) = e^{\left(\int_{t_0}^{t} f(\tau)\,d\tau + C + \ln(y_0)\right)}, \quad \text{Let } L = e^{C+\ln(y_0)}$$

$$\Rightarrow \boxed{y(t) = L\,e^{\int_{t_0}^{t} f(\tau)\,d\tau}}$$

With the initial condition $y(t_0) = y_0$, we find

$$y_0 = y(t_0) = L\,e^{\int_{t_0}^{t_0} f(\tau)\,d\tau} = L$$

$$\Rightarrow y(t) = y_0\,e^{\int_{t_0}^{t} f(\tau)\,d\tau}.$$

ii) $y'(t) = f(t)\,y(t) + h(t)$ (Linear inhomogeneous ODE)

Technique: Variation of constants (J. Bernoulli 1697)

This technique is based on the fact that the solution of the inhomogeneous ODE has the same structure as for homogeneous ODEs but with $L = L(t)$.

Write $y(t) = L(t)\,e^{\int_{t_0}^{t} f(\tau)\,d\tau}$ with $R(t) = e^{\int_{t_0}^{t} f(\tau)\,ds}$.

Then $y'(t) = L'(t)R(t) + L(t)R'(t)$, with

$$R'(t) = \frac{d}{dt}\left(e^{\int_{t_0}^{t}\beta(\tau)d\tau}\right) = \beta(t)\, e^{\int_{t_0}^{t}\beta(\tau)d\tau} = \beta(t)R(t).$$

Hence $L(t)R'(t) = \beta(t)\big(L(t)R(t)\big) = \beta(t)\,y(t)$, and therefore

$$y'(t) = \beta(t)y(t) + L'(t)R(t)$$

$$\Leftrightarrow \quad L'(t)R(t) = h(t)$$

$$\Rightarrow \quad L(t) = \int_{t_0}^{t} h(\tau)/_{R(\tau)}\, d\tau + C$$

$$\Rightarrow \quad y(t) = \left(\int_{t_0}^{t} \underbrace{\left(e^{-\int_{\tau_0}^{\tau}\beta(\tilde{\tau})d\tilde{\tau}}\right)} h(\tau)\,d\tau + C\right) e^{\int_{t_0}^{t}\beta(\tau)d\tau} .$$

This is called the integrating factor of the ODE $y'(t)+\beta(t)y(t)=h(t)$.

Ex: $y'(t) - \sin(t)y(t) = 1$.

Step 1: Solve for the homogeneous ODE $y'(t) - \sin(t)y(t) = 0$.

$$\Rightarrow \quad \frac{1}{y(t)}\, y'(t) = \sin(t)$$

$$\Rightarrow \int_{y(t_0)}^{y(t)} \frac{1}{z}\, dz = \int_{t_0}^{t} \sin(\tau)\,d\tau \quad \Rightarrow \ln(y(t)) = -\cos(t) + C$$

$$\Rightarrow y(t) = L\, e^{-\cos(t)}$$

Step 2: Let $y(t) = L(t)e^{-\cos t}$ be the solution to the inhomog. ODE.

$$y'(t) = L'(t)e^{-\cos t} + L(t)e^{-\cos t}\sin(t) = L'(t)e^{-\cos t} + \sin(t)\,y(t)$$

Thus $y'(t) - \sin(t) y(t) = L'(t) e^{-\cos t}$

Step 3 : Plug in the ODE.

$$\Rightarrow L'(t) e^{-\cos t} = 1$$

$$\Rightarrow L(t) = \int_{t_0}^{t} e^{\cos \tau} \, d\tau + C$$

$$\Rightarrow y(t) = \left(\int_{t_0}^{t} e^{\cos \tau} \, d\tau + C \right) e^{-\cos t} .$$

② $P(t, y(t)) + Q(t, y(t)) y'(t) = 0$.

- Suppose that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial t}$. The idea then to solve this ODE is to find a potential function $V(t, y(t))$ such that $\frac{\partial V}{\partial t} = P$ and $\frac{\partial V}{\partial y} = Q$. In that case, $\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial V}{\partial y}\right)$ (mixed partial derivatives are equal), and in that case $0 = P + Q y' \iff \underbrace{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial y} y'(t) = \frac{d}{dt}\left(V(t, y(t))\right)}_{\text{This is the multivariable chain rule !}} = 0$.

$$\Rightarrow V(t, y(t)) = \text{constant}$$

↳ Solve for $y(t)$ ...

- What happens when $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial t}$? Find a function M such that $\frac{\partial (M(t,y(t)) P(t,y(t)))}{\partial y} = \frac{\partial}{\partial t} (M(t,y(t)) Q(t,y(t))$ and multiply the ODE by M:

$$\Rightarrow \underbrace{M(t,y(t))P(t,y(t))}_{\tilde{P}} + \underbrace{M(t,y(t))Q(t,y(t))}_{\hat{Q}}\, y'(t) = 0 \qquad \text{with } \frac{\partial \hat{P}}{\partial y} = \frac{\partial \hat{Q}}{\partial t} \ldots$$

§1.2.2. Second order ODEs

We will consider three types of second order ODEs

① $y''(t) + a(t)y'(t) + b(t)y(t) = 0$ (Linear and homogeneous)

② $y''(t) = f(y'(t), y(t))$, f given (Autonomous ODE)

③ $y''(t) = f(y'(t), t)$, f given (No explicit y-dependence)

① $y''(t) + a(t)y'(t) + b(t)y(t) = 0$.

Idea: Write $y(t) = e^{\left(\int_{t_0}^{t} p(s)\,ds\right)}$ and find the function p that solves the ODE.

$$\Rightarrow y'(t) = e^{\left(\int_{t_0}^{t} p(s)\,ds\right)} p(t) = p(t)y(t)$$

$$\Rightarrow y''(t) = p'(t)y(t) + p(t)y'(t) = p'(t)y(t) + (p(t))^2 y(t)$$

Substitute in the ODE:

$$(p'(t) + (p(t))^2)y(t) + a(t)p(t)y(t) + b(t)y(t) = 0$$

$$\Leftrightarrow \boxed{p'(t) + (p(t))^2 + a(t)p(t) + b(t) = 0} \qquad \left(\begin{matrix}\text{Riccati } 1723\\ \text{Euler } 1728\end{matrix}\right)$$

⊕ We transformed the 2^{nd} order ODE → 1^{st} order ODE

⊖ The new ODE is nonlinear and inhomogeneous ...

② $y''(t) = f(y'(t), y(t))$, f is given (autonomous ODE)

<u>Idea</u>: Express y' in terms of y ...

↳ Let $y'(t) = \varphi(y(t))$ and find what φ is ...

$\Rightarrow y''(t) = \varphi'(y(t)) \cdot y'(t)$. But $y'(t) = \varphi(y(t))$ and $y''(t) = f(\varphi(y(t)), y(t))$

$$\Rightarrow \boxed{\varphi'(y(t)) \cdot \varphi(y(t)) = f(\varphi(y(t)), y(t))}.$$

$\Rightarrow$ 1^{st} order ODE in terms of y $(\varphi'(y) \cdot \varphi(y) = f(\varphi(y), y))$

- Find φ
- Once φ is found, solve $y'(t) = \varphi(y(t))$ for $y(t)$.

③ $y''(t) = f(y'(t), t)$ (no explicit y-dependence)

<u>Idea</u>: Let $y'(t) = \varphi(t)$ $\Rightarrow$ $y''(t) = \varphi'(t)$

We then get $\varphi'(t) = f(\varphi(t), t)$ (1^{st} order ODE).

- Find φ
- Once φ is found, we get $y(t) = \displaystyle\int_{t_0}^{t} \varphi(\xi)\,d\xi + C$.

Summary :

1) 1^{st} order ODEs

- In particular for linear ODEs
  - Find solution to the homogeneous ODE
  - Find solution to the inhomogeneous ODE using the variation of constants technique.
- For nonlinear ODEs, use, for example, integration in t.

2) 2^{nd} order ODEs : Transform into 1^{st} order ODEs.

Remark : Leonhard Euler wrote > 800 pages on ODEs solutions
Erich Kamke (1942) : 1500 DE and solutions
Joseph Liouville (1841) : Certain ODEs cannot be solved in terms of elementary functions

e.g.: $y'(t) = y(t)^2 + t^2$.

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§ 1.3. Series solutions

Idea: Represent the solution of an ODE (typically a 2^{nd} order ODE) by a Taylor series:

$$\boxed{y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k}.$$

(The coefficients of the series are given by $\{a_k\}_{k=0}^{\infty}$, and x_0 is called the point of expansion of the series!)

Some facts about series:

i) A series $y(x) = \sum_{k=0}^{m} a_k (x-x_0)^k$ is said to converge at a point x if

$$y(x) = \lim_{m\to\infty} \left(\sum_{k=0}^{m} a_k (x-x_0)^k \right)$$

exists for that x.

It is absolutely convergent at a point x if

$$\lim_{m\to\infty} \left(\sum_{k=0}^{m} |a_k| \, |x-x_0|^k \right)$$

exists for that x.

ii) (Ratio test) The series $y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ converges absolutely (at x) if

$$\lim_{k\to\infty} \left| \frac{a_{k+1}\,(x-x_0)^{k+1}}{a_k\,(x-x_0)} \right| < 1$$

$$\Longleftrightarrow \quad |x-x_0| \left(\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right) < 1.$$

- If $|x-x_0| \left(\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right) > 1$, then $y(x)$ diverges.
- If $|x-x_0| \left(\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right) = 1$, then the test is inconclusive.

iii) Note that if x and x_0 are such that $|x-x_0| < \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$, then the series $y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ will be absolutely convergent.

We call the number $\delta \equiv \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$ the radius of convergence of the series.

This means that for $x \in B_\delta(x_0) := \{ \tilde{x} : |\hat{x} - x_0| < \delta \}$, the series will converge absolutely.

iv) The function $x \mapsto y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ is continuous and has derivatives of all order whenever $|x-x_0| < \delta$ (= radius of convergence). The derivatives can be obtained by differentiating the series term by term:

$$y'(x) = \sum_{k=0}^{\infty} a_k \frac{d}{dx}(x-x_0)^k = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}.$$

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We will consider the 2nd-order homogeneous ODE

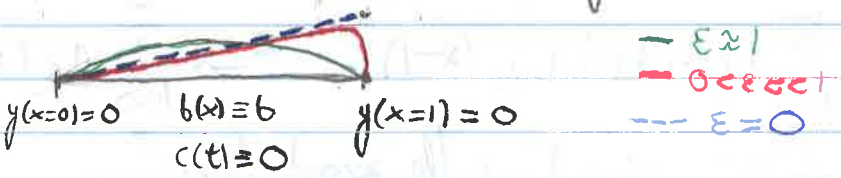
$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0.$$

These ODEs occur in many applications.

Examples: 1) Diffusion-advection-reaction problems (1D, stationary)

$$\varepsilon y''(x) + b(x) y'(x) + c(x) = 0, \text{ where } 0 < \underset{\uparrow \text{ Diffusivity}}{\varepsilon} << 1 \text{ and } \underset{\nwarrow \text{ Transport speed}}{b(x)} \geq 0$$

The solution does <u>not</u> depend continuously on ε as $\varepsilon \to 0$...



2) The Bessel equation

$$x^2 y''(x) + x y'(x) + (x^2 - n^2) y(x) = 0,$$ where n is non-negative integer.

↳ Arise in wave phenomena (fluid dynamics, optics, electromagnetism).

3) The associated Laguerre equation

$$x y''(x) + (\alpha + 1 - x) y'(x) + n y = 0,$$ where α is a real number and n is a non-negative integer.

↳ Arise in quantum mechanics as the radial component of the wave function associated to the hydrogen atom.

§1.3.1 $P(t_0) \neq 0$ (reg)

4) The Airy's equation

$$y''(x) - x y(x) = 0.$$ (Arise in quantum mechanics, astronomy, optics...)

⇒ Let's attempt to find its series solution about $x_0 = 1$.

↳ $y(x) = \sum_{k=1}^{\infty} a_k (x-1)^k$ and compute, if possible, the coefficients $\{a_k\}_{k=1}^{\infty}$.

<u>Idea</u>: Derive recurrence relations for the coefficients $\{a_k\}_{k=1}^{\infty}$.

- $$y'(x) = \sum_{k=1}^{\infty} k a_k (x-1)^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x-1)^k$$
- $$y''(x) = \sum_{k=1}^{\infty} (k+1) k a_{k+1} (x-1)^{k-1} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} (x-1)^k$$

- Plug into ODE $y''(x) - xy = 0$:

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2} (x-1)^k - x \sum_{k=0}^{\infty} a_k (x-1)^k . \qquad (\underline{*})$$

Write $x = (x-1)+1$ in the second term $\nearrow$:

$$x \sum_{k=0}^{\infty} a_k (x-1)^k = \left[(x-1)+1\right] \sum_{k=0}^{\infty} a_k (x-1)^k$$

$$= \sum_{k=0}^{\infty} a_k (x-1)^{k+1} + \sum_{k=0}^{\infty} a_k (x-1)^k$$

$$= \sum_{k=1}^{\infty} a_{k-1} (x-1)^k + \sum_{k=0}^{\infty} a_k (x-1)^k \leftarrow$$

$$\text{Then } (\underline{*}) = \underbrace{\left(\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - a_k\right](x-1)^k\right)}_{= 2a_2 - a_0 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - a_k\right](x-1)^k} - \left(\sum_{k=1}^{\infty} a_{k-1} (x-1)^k\right)$$

$$\Rightarrow (*) = \sum_{k=1}^{\infty} \left[(k+2)(k+1) a_{k+2} - a_k - a_{k-1}\right](x-1)^k + (2a_2 - a_0)$$

$$\equiv 0 .$$

In order for this to be equal to 0, we require

$$\begin{cases} 2a_2 - a_0 = 0 \\ (k+2)(k+1) a_{k+2} - a_k - a_{k-1} = 0 \quad \forall k \geq 1 . \end{cases}$$

We find $a_2 = \frac{a_0}{2}$, a_0 is a free parameter

a_1 is also a free parameter (no eq. for a_1)

$a_3 = (a_0 + a_1)/6$ $(k=1)$

$a_4 = (a_1 + a_2)/12 = a_1/12 + a_0/24$ $(k=2)$

We get
$$y(x) = a_0\left(1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\dots\right)$$
$$+ a_1\left((x-1)+\frac{(x-1)^2}{6}+\frac{(x-1)^3}{12}+\dots\right)$$

Summary:
1) Write the solution in series form.
2) Plug into the ODE
3) Order the terms and find recurrence relations.

This process hold in general for ODEs of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \text{ (where P is continuous)}$$

at any point $x \equiv x_0$ for which $P(x_0) \neq 0$ and we call such a point a regular point of P. Indeed, if P is continuous, there exists a number $\varepsilon > 0$ such that $P(x) \neq 0$ for every $x \in B_\varepsilon(x_0)$ (sufficiently close to x_0.

§1.3.2 : The case where $P(x_0) = 0$ (x_0 is called a singular point of P).

- Consider the ODE (Euler, 1769):

$$\underbrace{x^2(a+bx)}_{P(x)}y''(x) + \underbrace{x(c+dx)}_{Q(x)}y'(x) + \underbrace{(f+gx)}_{R(x)}y(x) = 0.$$

Here, a, b, c, d, e, and f are given real numbers.

$\Rightarrow$ Use the series solutions $y(x) = x^q \sum_{k=0}^{\infty} a_k x^k$, for some $q \in \mathbb{R}$ at the point $x_0 = 0$.

e.g.,: $ax^2y''(x) + cxy'(x) + f y(x) = 0$

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Inserting this into the ODE (*) and comparing the coefficients (after ordering the terms), we get the recurrence relations

Recurrence (R)

$$\begin{cases} \cdot \left(q(q-1)a + qc + \beta\right) a_0 = 0 & (k=0) \\ \cdot \left((q+k)(q+k-1)a + (q+k)c + \beta\right) a_k \text{ oc} & \\ \quad = -\left((q+k-1)(q+k-2)b + (q+k-1)d + g\right) a_{k-1} . & (k \geq 1) \end{cases}$$

We need to have $a_0 \neq 0$ (otherwise $a_k = 0 \ \forall k \geq 0$).

Therefore, the quantity of interest is the index equation:

$$\boxed{\begin{aligned} \chi(q) &= q(q-1)a + qc + \beta &= 0 \\ &= q^2 - (a-c)q + \beta &= 0 \end{aligned}}$$

Solving $\chi(q) = 0$ $(a_0 \neq 0)$ is a quadratic equation that has two roots $q_1, q_2 \in \mathbb{R}$. (Note: Recurrence relation $\begin{cases} \chi(q)\, a_0 = 0 \\ \chi(q+k)\, a_k = -[(q+k-1)(q+k-2)b + (q+k-1)d + g]a_{k-1} \end{cases}$

In what follows, we will assume without loss of generality that $q_1 \geq q_2$ (or $\mathrm{Re}\, q_1 \geq \mathrm{Re}\, q_2$). There are three cases:

① $q_1 \neq q_2$, $q_1 - q_2 \notin \mathbb{N}$.

$\Rightarrow$ $q_2 + k \neq q_1$ for any $k \in \mathbb{N}$, and as a result

$\chi(q_2 + k) \neq 0$ and $\chi(q_1 + k) \neq 0$ for any $k \in \mathbb{N}$.

$\Rightarrow$ The recurrence (R) is well-defined and we can proceed as for the regular point.

② $q_1 \neq q_2$, $q_1 - q_2 \in \mathbb{N}$.

In that case, we get $\mathcal{X}(q_1+k) \neq 0$ for any $k \geq 1$ => recurrence works

but $\mathcal{X}(q_2+k) = 0$ for $k = q_1 - q_2 \in \mathbb{N}$, and at that point the recurrence stops.

We will not cover the method to find the second general solution. (see, e.g., ...)

③ $q_1 = q_2$ (double root)

Here, the recurrence iterations for q_1 and q_2 are the same, and so we get one solution (a 2nd order ODE needs two...)

How do we get this second solution? Using Frobenius method.

Case ① : $q_1 \neq q_2$, $q_1 - q_2 \notin \mathbb{N}$

Method : i) Fix $a_0 = 1$ and iterate the recurrence (R) for q_1. This will give coefficients $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \dots$ => $y_1(x) = x^{q_1} \sum_{k=0}^{\infty} a_k^{(1)} x^k$.

ii) Again, fix $a_0 = 1$ and iterate the recurrence (R) with q_2. This will give

=> $a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \dots$ => $y_2(x) = x^{q_2} \sum_{k=0}^{\infty} a_k^{(2)} x^k$.

iii) General solution of the ODE: Superposition principle

Given two solutions y_1 and y_2 of a linear homogeneous ODE, then any linear combination is a solution.

$$\Rightarrow y_{\text{general}}(x) = \alpha y_1(x) + \beta y_2(x), \quad \alpha, \beta \in \mathbb{R}$$

is also a solution.

Indeed, let

$$\textcircled{I} : a_n(x) y_1^{(n)}(x) + \dots + a_1(x) y_1'(x) + a_0(x) y_1(x) = 0$$

$$\textcircled{II} : a_n(x) y_2^{(n)}(x) + \dots + a_1(x) y_1'(x) + a_0(x) y_1(x) = 0$$

Then $\alpha \textcircled{I} + \beta \textcircled{II}$ gives

$$a_n(x)(\alpha y_1^{(n)}(x) + \beta y_2^{(n)}(x)) + \dots + a_1(x)(\alpha y_1^{(1)}(x) + \beta y_2^{(1)}(x)) + a_0(x)(\alpha y_1(x) + \beta y_2(x)) = 0,$$

$$\Leftrightarrow a_n(x)(\alpha y_1 + \beta y_2)^{(n)}(x) + \dots + a_1(x)(\alpha y_1 + \beta y_2)^{(1)}(x) + a_0(x)(\alpha y_1 + \beta y_2)(x) = 0$$

$$\Rightarrow \alpha y_1(x) + \beta y_2(x) \text{ is a solution}$$

In our case, the general solution formed from the series solution is

$$y(x) = C_1 x^{q_1} \sum_{k=0}^{\infty} a_k^{(1)} x^k + C_2 x^{q_2} \sum_{k=0}^{\infty} a_k^{(2)} x^k \quad \text{with } C_1, C_2 \in \mathbb{R}.$$

Lecture 4

(*)

Re-write recurrence relation! + ODE

Ex: $2x^2 y''(x) - x y'(x) + (-x+1) y(x) = 0$

$$\Rightarrow x^2(\underset{\substack{\parallel\\2}}{a} + \underset{\substack{\parallel\\0}}{b}x) y''(x) + x(\underset{\substack{\parallel\\-1}}{c} + \underset{\substack{\parallel\\0}}{d}x) + (\underset{\substack{\parallel\\1}}{f} + \underset{\substack{\parallel\\-1}}{g}x) y(x) = 0$$

The inder equation is then $\chi(q) = q^2 - 3q + 1 = (q-1)(q-1/2)$

$$\Rightarrow q_1 = 1, \; q_2 = 1/2, \; q_1 - q_2 = 1/2 \notin \mathbb{N}. \quad (\text{Case } ①)$$

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i) Recurrence with $q_1 = 1$:

$$(R) : \mathcal{X}(q_1 + k)a_k = a_{k-1}, \quad k \geq 0 \text{ with } a_0 = 1$$

So $$a_k = \frac{a_{k-1}}{\mathcal{X}(q_1+k)} = \frac{a_{k-1}}{2(q_1-1+k)(q_1-\frac{1}{2}+k)} \overset{q_1=1}{=} \frac{a_{k-1}}{2k(k+1/2)} = \frac{a_{k-2}}{2^2 k(k-1)(k+\frac{1}{2})(k-\frac{1}{2})}$$

So $$a_1^{(1)} = \frac{1}{2 \cdot 1 \cdot (1+1/2)} = \frac{1}{3}, \quad a_2^{(1)} = \frac{1/3}{2 \cdot 2(2+1/2)} = \frac{1}{30},$$

$$a_k^{(1)} = \frac{1}{2^k\left(\frac{2}{2}\cdot\frac{3}{2}\right)\left(\frac{4}{2}\cdot\frac{5}{2}\right)\left(\frac{6}{2}\frac{7}{2}\right)\cdots\left(\frac{2k}{2}\frac{2k+1}{2}\right)} = \frac{1}{2^k 2^{-2k}(2k+1)!}$$

$$= \frac{2^k}{(2k+1)!}$$

$$\Rightarrow y_1(x) = x\sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!} x^k$$ (and its radius of convergence is $\delta = \infty$).

ii) Recurrence with $q_2 = 1/2$: $\mathcal{X}(q_2 + k)a_k = a_{k-1} \Leftrightarrow a_k = \frac{a_{k-1}}{2(k-\frac{1}{2})k}$

$$\Rightarrow a_k^{(2)} = \frac{2^k}{(2k)!}$$

$$\Rightarrow y_2(x) = \sqrt{x}\sum_{k=0}^{\infty} \frac{2^k}{(2k)!} x^k$$

iii) Superposition principle (General solution):

$$y(x) = C_1 x \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!} x^k + C_2 \sqrt{x} \sum_{k=0}^{\infty} \frac{2^k}{(2k)!} x^k.$$

//

Case ③ : The roots q_1 and q_2 of the index equation

$$\chi(q) = q(q-1)a + cq + \beta = 0$$

are the same $(q_1 = q_2)$.

- We will focus on the ODE

$$x^2 y''(x) - x y'(x) + (1-x)\, y(x) = 0 \qquad (*)$$

with $\chi(q) = (q-1)^2 \Rightarrow$ two zeros $q_1 = q_2 = 1$.

- The recurrence relation $(a_0 = 1)$ is

$$\chi(q_1+k)\, a_k = a_{k-1}, \quad \text{with } \chi(q_1+k) = (q_1+k-1)^2 = (1+k-1)^2 = k^2.$$

$$\Rightarrow a_k = a_{k-1}/k^2.$$

$$\Rightarrow a_k = \frac{1}{1}\cdot\frac{1}{2^2}\cdot\frac{1}{3^2}\cdot\ldots\cdot\frac{1}{k^2} = \frac{1}{(k!)^2} \Rightarrow y(x) = x\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}.$$

- This gives one solution. Alas, we need two!
- Second solution: Use Frobenius method.

Idea: Do not compute the coefficients for some specific q, but for an arbitrary one.

Here, the recurrence relation is $\chi(q+k)\, a_k(q) = a_{k-1}(q)$ for $k \geq 1$. ($a_0 = 1$, and now the coefficients depend on q.)

Define: $z(x,q) := x^q\left(\sum_{k=0}^{\infty} a_k(q)\, x^k\right)$ and plug this into the ODE $(*)$:

Then $\frac{\partial z}{\partial x}(x,q) = q x^{q-1}\left(\sum_{k=0}^{\infty} a_k(q)\, x^k\right) + x^q \sum_{k=0}^{\infty} a_k(q)\, k\, x^{k-1}$.

$$= x^{q-1}\left[\sum_{k=0}^{\infty} (k+q)\, a_k(q)\, x^k\right]$$

and similarly, $\frac{\partial^2 z}{\partial x^2}(x,q) = x^{q-2}\left[\sum_{k=0}^{\infty} (k+q)(k+q-1)\, a_k(q)\, x^k\right]$.

Now substitute $z(x,q)$ for $y(x)$ into the ODE:

$$\begin{aligned} \mathrm{ODE}(z(x,q)) = {} & x^q\left(\sum_{k=0}^{\infty}(k+q)(k+q-1)\,a_k(q)\,x^k\right) \\ & - x^q\left(\sum_{k=0}^{\infty}(k+q)\,a_k(q)\,x^k\right) \\ & + x^q\left(\sum_{k=0}^{\infty} a_k(q)\,x^q\right) - \overbrace{x^{q+1}\left(\sum_{k=0}^{\infty} a_k(q)\,x^k\right)}^{} \\ & \qquad = x^q \sum_{k=1}^{\infty} a_{k-1}(q)\,x^k \end{aligned}$$

$$\Rightarrow \mathrm{ODE}(z(x,q)) = x^q\left[\sum_{k=0}^{\infty} \underbrace{\left(\underbrace{\left((q+k)(q+k-1)-(q+k)+1\right)}_{\mathcal{X}(q+k)} a_k(q) - a_{k-1}(q)\right)}_{\mathcal{X}(q+k)\,a_k(q) - a_{k-1}(q) \,=\, 0 \text{ by design (recurrence relation)}} x^k\right]$$

$$+ x^q\big(\underbrace{q(q-1)-q+1}_{\mathcal{X}(q)}\big)$$

$\therefore$ $\mathrm{ODE}(z(x,q)) = x^q\,\mathcal{X}(q)$, which is zero when setting $q \equiv q_1 = q_2 = 1$.

No surprise here, this is the solution we got from the recurrence relation ...

To obtain a second solution, differentiate both sides of ODE$(z(x,q))$ with respect to q: On the one hand, we have

$$\frac{\partial}{\partial q}\left(x^q \mathcal{X}(q)\right) = \ln x \, x^{q} \mathcal{X}(q) + x^q \mathcal{X}'(q),$$

with $\mathcal{X}(q) = (q-1)$ and $\mathcal{X}'(q) = 2(q-1)$, so that

$$\left.\frac{\partial}{\partial q}\left(x^q \mathcal{X}(q)\right)\right|_{q=q_1=1} = \ln x \, x^{1}(1-1)^2 + x^1 \cdot 2(1-1) \equiv 0.$$

$$\therefore \boxed{\left.\frac{\partial}{\partial q}\left(x^q \mathcal{X}(q)\right)\right|_{q=q_1=1} = 0}$$

On the other hand,

$$\left.\frac{\partial}{\partial q}\left[x^2 \frac{\partial^2 z}{\partial x^2}(x,q) - x\frac{\partial z}{\partial q}(x,q) + (1-x)z(x,q)\right]\right|_{q=q_1=0} = 0 \quad \text{by this}$$

$$\Leftrightarrow \left.\left[x^2 \frac{\partial^3 z}{\partial x^2 \partial q}(x,q) - x\frac{\partial^2 z}{\partial x \partial q}(x,q) + (1-x)\frac{\partial z}{\partial q}(x,q)\right]\right|_{q=q_1=1} = 0$$

$$\Leftrightarrow \left[x^2 \underbrace{\frac{\partial^2}{\partial x^2}\left(\frac{\partial z(x,q)}{\partial q}\right)}_{y_2''(x)} - x \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial q}(x,q)\right)}_{y_2'(x)} + (1-x)\underbrace{\frac{\partial z}{\partial q}(x,q)}_{y_2(x)}\right]_{q=q_1=1} = 0$$

$$\therefore \; y_2(x) := \left.\frac{\partial z(x,q)}{\partial q}\right|_{q=q_1=1} \quad \text{is a second solution!}$$

Now, $\frac{\partial z}{\partial q}(x,q) = \frac{\partial}{\partial q}\left(x^q \sum_{k=0}^{\infty} a_k(q)\, x^k\right)$

$$= \ln x \left(x^q \sum_{k=0}^{\infty} a_k(q)\, x^k\right) + x^q \sum_{k=0}^{\infty} a_k'(q)\, x^k .$$

$\therefore\ y_2(x) = \frac{\partial z}{\partial q}(x,q)\Big|_{q=q_1=1}$

$$= x\Big(\sum_{k=0}^{\infty} \underbrace{a_k'(q)\Big|_{q=q_1=1}}_{=\,???} x^k\Big) + x\ln x\Big(\underbrace{\sum_{k=0}^{\infty} a_k(q_1)\, x^k}_{\text{known from the first recurrence relation}}\Big)$$

How do we determine $\{a_k'(q)\big|_{q=q_1=1}\}_{k=0}^{\infty}$? (Compute $a_k(q)$ and differentiate ...

$\Rightarrow \mathcal{X}(q+k)\cdot a_k(q) = a_{k-1}(q)$ with $k \geq 1$, $\mathcal{X}(q+k) = (q+k-1)^2$

$$\Rightarrow a_k(q) = \frac{a_{k-1}(q)}{(q+k-1)^2}$$

$\Rightarrow a_k(q) = \frac{1}{q^2}\cdot\frac{1}{(q+1)^2}\cdot\ldots\cdot\frac{1}{(q+k-1)^2}$ (and note $a_k(q_1=1) = \frac{1}{(k!)^2}$.)

$$\Rightarrow \ln(a_k(q)) = \ln\left(\frac{1}{q^2}\right) + \ln\left(\frac{1}{(q+1)^2}\right) + \ldots + \ln\left(\frac{1}{(q+k-1)^2}\right)$$

$$\Rightarrow \frac{d}{dq}\left(\ln(a_k(q))\right) = \frac{a_k'(q)}{a_k(q)} = -2\left[\sum_{j=1}^{k} \frac{1}{(q+j-1)}\right]$$

Hence $a'_k(q_1=1) = -2a_k(q_1=1)\left(\sum_{j=1}^{k} 1/j\right)$

$$= -\frac{2}{(k!)^2}\left(\sum_{j=1}^{k} 1/j\right)$$

$$\therefore\ y_2(x) = -2x\sum_{k=0}^{\infty}\left(\frac{1}{(k!)^2}\left(\sum_{j=1}^{k} 1/j\right)x^k\right) + x\ln x\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}.$$

Finally, the general solution is $y(x) = C_1 y_1(x) + C_2 y_2(x)$.

Phew!

§1.4 Graphical solutions

Idea: Pictures are often more useful than formulas for analyzing nonlinear (autonomous) ODEs. To this end, we shall interpret a differential equation as a vector field.

Ex: Consider the nonlinear ODE $\dot{x} = \sin x$ (remember: $\dot{x} = \frac{dx(t)}{dt}$)
$x(0) = \pi/4$.

Question: What are the qualitative features of this ODE?
What is $\lim_{t\to\infty} x(t) = ?$

Let's solve the ODE using separation of variables:

$$\int_{\pi/4}^{x(t)} \frac{1}{\sin z}\,dz = t$$

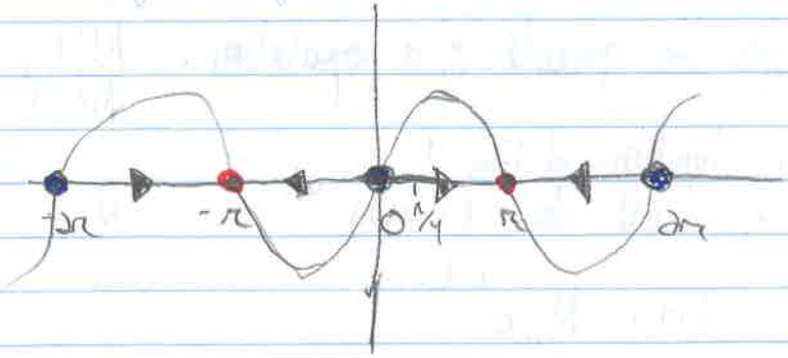
$$= -\ln|\csc x + \cot x|$$

$$\Rightarrow \quad t = \ln\left|\frac{\csc(\pi/4) + \cos(\pi/4)}{\csc(x(t) + \cot(x(t))}\right| . \quad \Rightarrow x(t) = 2\arctan\left(\frac{e^{t}}{1+\sqrt{2}}\right).$$

$\Rightarrow$ We have an exact (implicit) solution ...
... but it looks complicated and difficult to interpret ...
e.g., what is $\lim_{t\to\infty} x(t) = ?!$

- Let's think geometrically instead and do a graphical analysis of the ODE by plotting the graph of $\dot{x}(t)$ vs x ! Remember: $\dot{x} = \sin(x(t))$, so we are really plotting $\sin x$ vs x

● : Stable fixed point
● : Unstable fixed point



Thinking of t as time, x as the position of an imaginary particle, and $\dot{x}$ as the velocity of the imaginary particle. The the ODE $\dot{x}(t) = \sin(x(t))$ represents a vector field on the line: it dictates the velocity $\dot{x}$ at each x.

Draw arrows on the plot $\dot{x}$ vs x to indicate the corresponding velocity $\dot{x}$ at each x. (Think of it as a flow)

Here, a right arrow ➡ means $\dot{x} > 0$ (the particle moves to the right)
a left arrow ⬅ means $\dot{x} < 0$ (the particle moves to the left)

The points at which $\dot{x} = 0$ are called fixed points.
● : Stable fixed pts (attractors or sinks)
● : Unstable fixed pts (repellers or sources)

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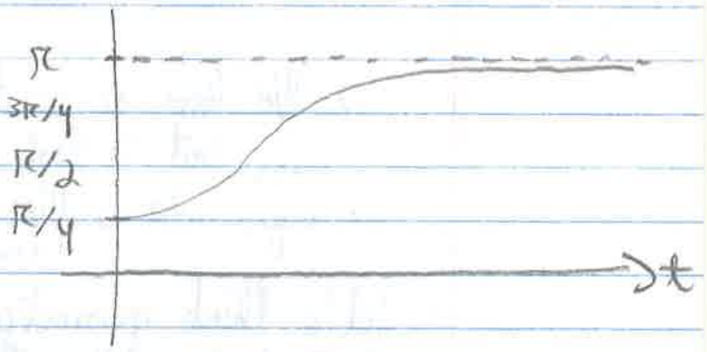
- So, given $x(0)=\pi/4$, what is $\lim_{t\to\infty} x(t) = ?$ Look at the graph, we observe $\lim_{t\to\infty} x(t) = \pi$, which is also mathematically (rigorously) correct.

Qualitative form of the solution:
(Notice the inflexion point at $\pi/2$)

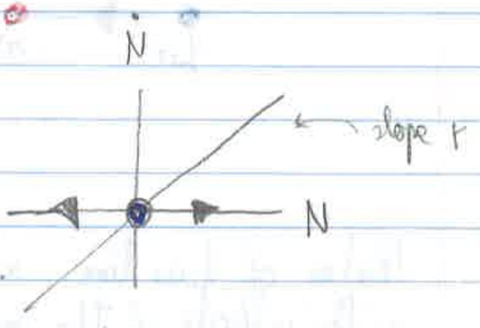


Pretty neat, eh?

Ex: (Population growth and the logistic equation)

Model for the growth of a population: $\dot{N}(t) = rN(t)$ with $N(0) = N_0 \ (>0)$

- $N(t)$: Population at time t
- r : Growth rate $(r>0)$

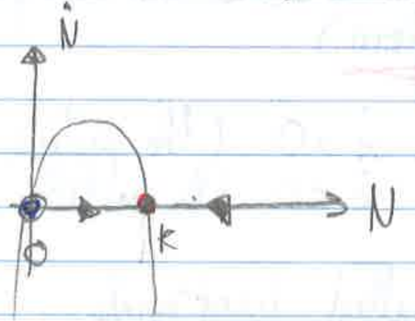


Solution: $N(t) = N_0 e^{rt}$

Get $\lim_{t\to\infty} N(t) = \infty$, which seems unrealistic...

Better model: $\dot{N}(t) = rN(t)\left(1-\frac{N(t)}{K}\right)$ (Verhulst, 1838)
K: Carrying capacity

This ODE has an exact solution (HW 2), but let's look at its graphical solution:



We see $\lim_{t\to\infty} N(t) = K$ ("carrying capacity")

Note: All this apply only to 1st-order autonomous ODEs $\dot{x}(t) = f(x(t))$.
Later we will see how to draw graphical solutions to system of ODEs.

§2. Linear ODEs

§2.1 Linear homogeneous ODE

§2.1.1 Examples

ⓐ $y'(t) + 2y(t) = 0$ (1st-order)

Solution: Write the ODE as $\frac{y'(t)}{y(t)} = -2$

$$\Rightarrow \int_{y(t_0)}^{y(t)} \frac{1}{z}\,dz = -2(t-t_0)$$

$$\Rightarrow \ln(y(t)) = -2t + C$$

$$\Rightarrow y(t) = e^{C}\cdot e^{-2t} = \underset{\nwarrow \text{ constante}}{L}e^{-2t}$$

ⓑ $y''(t) + 5y'(t) + 6y(t) = 0$ (2nd order)

Is the solution also an exponential?

Ansatz: $y(t) = e^{\lambda t}$ (Ansatz is a German word that means "initial placement of a tool at a work piece" – in our context it means "educated guess".)

Plug into ODE:

$$\Rightarrow (\lambda^2 + 5\lambda + 6)e^{\lambda t} = 0 \iff (\lambda^2 + 5\lambda + 6) = 0$$

$$\iff (\lambda + 2)(\lambda + 3) = 0$$

$$\iff \lambda = -2 \text{ or } -3$$

We have two solutions: $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$.

Superposition principle: $y(t) = C_1 y_1(t) + C_2 y_2(t)$
$= C_1 e^{-2t} + C_2 e^{-3t}$.

Suppose we have for initial conditions (I.C.) $y(0)=2$, $y'(0)=3$.

$\Rightarrow$ $y(t)=C_1e^{-2t}+C_2e^{-3t}$ and $y'(t)=-2C_1e^{-2t}-3C_2e^{-3t}$

Then $y(0)=C_1+C_2=2$ and $y'(0)=-2C_1-3C_2=3$

$$\Rightarrow \begin{pmatrix}1 & 1\\ -2 & -3\end{pmatrix}\cdot\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}2\\ 3\end{pmatrix} \quad \text{(Linear algebra)}$$

Gauss elimination :
$$\left(\begin{array}{cc|c}1 & 1 & 2\\ -1 & -3/2 & 3/2\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & -1/2 & 7/2\end{array}\right)$$
$$\longrightarrow\left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & 1 & -7\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1 & 0 & 9\\ 0 & 1 & -7\end{array}\right)$$

$\Rightarrow$ $C_1=9$, $C_2=-7$.

$$\Rightarrow \quad y(t)=9e^{-2t}-7e^{-3t}.$$

Ⓒ In general : $y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\dots+a_1y'(t)+a_0y(t)=0$.
(n^{th}-order homogeneous ODE)

Ansatz : $y(t)=e^{\lambda t}$, where $y'(t)=\lambda e^{\lambda t}$, $y''(t)=\lambda^2e^{\lambda t}$, ..., $y^{(n)}(t)=\lambda^ne^{\lambda t}$

Substitute in the ODE to obtain :

$$\underbrace{(\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0)}\,e^{\lambda t}=0$$

$=p(\lambda)$, which is a polynomial of degree n. (Characteristic polynomial)

$\Rightarrow$ n roots

$\Rightarrow$ If we have n solutions : $y_1(t)=e^{\lambda_1 t}$, ... $y_n(t)=e^{\lambda_n t}$ ...

Question : Have we found all solutions? (Can any solution of the ODE Ⓒ be written as $y(t)=C_1e^{\lambda_1 t}+\dots+C_ne^{\lambda_n t}$?

Answer: Sometimes yes, sometimes no... We will see that if $\lambda_i = \lambda_j$ for some $i \neq j$, we will need to add "extra" solutions.

§ 2.1.2 Review of linear algebra I

Defn: Consider a set of functions $\{f_1, f_2, \dots, f_n\}$

a) Then an expression of the form

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_n f_n(t)$$

with $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ (or $\mathbb{C}$) is called a linear combination of $f_1, f_2, \dots, f_n$.

b) The set of all linear combinations of $\{f_1, f_2, \dots, f_n\}$ is called the span of $\{f_1, f_2, \dots, f_n\}$ and is denoted by

$$\text{Span}(\{f_1, f_2, \dots, f_n\}) := \Big\{ \underbrace{\sum_{i=1}^{n} \alpha_i f_i}_{\text{Elements of this form}} \ \underset{\text{such that}}{\Big|} \ \underbrace{\alpha_i \in \mathbb{R} (\text{or } \mathbb{C}) \text{ for } i \in \{1, 2, \dots, n\}}_{\text{these conditions hold}} \Big\}$$

Ex: $f_1(t) = 1+t$, $f_2(t) = e^{t}$, and $f_3(t) = e^{-t}$.

A possible linear combination is $3(1+t) - 5e^{t} + e^{-t}$.

$$\text{Span}(\{f_1, f_2, f_3\}) = \{\alpha_1(1+t) + \alpha_2 e^{t} + \alpha_3 e^{-t} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} (\text{or } \mathbb{C})\}.$$

Defn: We say that $f_1, f_2, \dots, f_n$ are linearly independent if none of them can be written as a linear combination of the others. Otherwise, they are linearly dependent.

Ex: The functions $h_1(t) = \sin t$, $h_2(t) = t \sin t$, and $h_3(t) = t$ are linearly independent.

Lemma: A set of functions $f_1, f_2, \dots, f_n$ are linearly independent if and only if in the situation that

$$(*) \qquad \alpha_1 f_1(t) + \alpha_2 f(t) + \dots + \alpha_n f(t) = 0$$

holds, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Proof: "$\Rightarrow$" Suppose $f_1, f_2, \dots, f_n$ are linearly independent.

(Proof by contradiction): Suppose $(*)$ is false. Then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ (or $\mathbb{C}$), not all zero, such that $\alpha_1 f_1(t) + \alpha_2 f_2(t) + \dots + \alpha_n f_n(t) = 0$. Suppose $\alpha_j \neq 0$ and write

$$\alpha_j f_j(t) = -\alpha_1 f_1(t) - \dots - \alpha_{j-1} f_{j-1}(t) - \alpha_{j+1} f_{j+1}(t) - \dots$$

$$\Rightarrow f_j(t) = -\frac{\alpha_1}{\alpha_j} f_1(t) - \dots - \frac{\alpha_{j-1}}{\alpha_j} f_{j-1}(t) - \frac{\alpha_{j+1}}{\alpha_j} f_{j+1}(t) - \dots$$

$\Rightarrow$ The functions $f_1, f_2, \dots, f_n$ are linearly dependent. Contradiction ↯.

"$\Leftarrow$" Let $(*)$ hold. If there exists f_j such that

$$f_j(t) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i f_i(t), \text{ then we would have}$$

$$\alpha_1 f_1(t) + \dots + (-1) f_i(t) + \dots + \alpha_n f_n(t) = 0,$$

which would contradict $(*)$. Hence $f_1, f_2, \dots, f_n$ are linearly independent. ■

Ex: a) Are $f_1(t) = t^2$ and $f_2(t) = t-1$ linearly independent?

Use the lemma: can we find α_1 and α_2 not both zero such that

$$\alpha_1 t^2 + \alpha_2 (t+1) = 0 \quad \text{for every } t \in \mathbb{R} \text{ ?}$$

A: No!

Differentiate twice w.r.t t to get $2\alpha_1 = 0 \Rightarrow \alpha_1 = 0$, and differentiate once and set $\alpha_1 = 0$ to get $\alpha_2 = 0$.

$\Rightarrow$ Only possible choice for every $t \in \mathbb{R}$ (or $\mathbb{C}$) is $\alpha_1 = \alpha_2 = 0$.

b) Are $g_1(t) = 1+t$, $g_2(t) = 1-t$, and $g_3(t) = 2t$ linearly independent?

Can we find α_1, α_2, and α_3 not all zero such that

$$\alpha_1(1+t) + \alpha_2(1-t) + \alpha_3(2t) = 0 \quad \text{for every } t \in \mathbb{R} \text{ ?}$$

$$\Rightarrow \quad (\alpha_1 - \alpha_2 + 2\alpha_3)t + (\alpha_1 + \alpha_2) = 0$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\text{Gauss elim.}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

$$\hookrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

$\hookrightarrow$ $\alpha_2 = \alpha_3$ and $\alpha_1 = -\alpha_2$.

So, choose α_1 to be anything non-zero to show linear dependence:

e.g., $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -1$, and

$$1 \cdot (1+t) - (1-t) - (2t) = 2t - 2t = 0 \quad \checkmark$$

Defn: Consider a set of functions $\{f_1, f_2, \dots, f_n\} =: S$

a) The maximum number of linearly independent functions in S (or equivalently, $\mathrm{Span}(S)$) is called the dimension of $\mathrm{Span}(S)$.

If all of $\{f_1, f_2, \dots, f_n\}$ are linearly independent, then

$$\dim(\mathrm{Span}(\{f_1, f_2, \dots, f_n\})) = n.$$

b) Let $k = \dim(\mathrm{Span}(S))$. Then $k \le n$ linearly independent functions $g_1, g_2, \dots, g_k \in \mathrm{Span}(S)$ are called a basis for $\mathrm{Span}(S)$.

Lemma: $\mathrm{Span}(\{g_1, \dots, g_k\}) = \mathrm{Span}(S)$.

Ex: let $S = \{1+t, 1-t, 2t, -5\}$. What is $\dim(\mathrm{Span}(S))$?

Note: $f_1 = \frac{f_3}{2} - \frac{f_4}{5}$ and $f_2 = -\frac{f_3}{2} - \frac{f_4}{5}$.

$\Rightarrow \mathrm{Dim}(\mathrm{Span}(S)) = 2$, and a basis is $\{1, t\}$.

§2.1.3 General solution of $\overbrace{y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = 0}^{(*)}$.

The characteristic polynomial of this ODE is $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$

Theorem: ⓐ Each solution of the ODE $(*)$ is infinitely (continuously) differentiable on $\mathbb{R}$ (for short, we say the solution is in $C^\infty(\mathbb{R})$).

ⓑ If y_1 and y_2 are solutions, so is $\alpha y_1 + \beta y_2$ (Superposition)

(I.e., the space of solutions is a vector space.)

ⓒ There exists n linearly independent solutions $\{y_1, y_2, \dots, y_n\}$ to the ODE (*).

Basis : $S_n = \{y_1, y_2, \dots, y_n\}$, and every solution to (*) is in Span(S_n), where dim set of solutions is n.

* * *

Corollary : Assume that the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the characteristic polynomial $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$ are distinct.

Then the functions $e^{\lambda_j t}$, $j = 1, \dots, n$, form a basis for the set of solutions.

Proof : Note that all $e^{\lambda_j t}$ solve the equations. If they are linearly independent, then the previous theorem shows they are a basis for the solution set.

=> We induct on n: Base case is $n = 1$.

$$Ce^{\lambda t} \neq 0 \quad \text{whenever} \quad C \neq 0 \quad \checkmark$$

=> Show for n :

$$\alpha_1 e^{\lambda_1 t} + \dots + \alpha_n e^{\lambda_n t} = 0 \quad ① \ (*)$$
$$\alpha_1 \lambda_1 e^{t} + \dots + \alpha_n \lambda_n e^{\lambda_n t} = 0 \quad ② \ d(*)/dt$$
$$\alpha_1 \lambda_n e^{t} + \dots + \alpha_n \lambda_n e^{\lambda_n t} = 0 \quad ③ \ \lambda_n(*)$$

Subtract $\lambda_n(*) - \frac{d}{dt}(*)$: $\alpha_1(\lambda_1 - \lambda_2)e^{\lambda_1 t} + \dots + \alpha_{n-1}(\lambda_1 - \lambda_{n-1})e^{\lambda_{n-1} t} = 0$

Induction step : $\alpha_i \underbrace{(\lambda_i - \lambda_n)}_{\neq 0} = 0 \implies \alpha_i = 0$ for $i = 1, \dots, n-1$.

Look back at (*): $\alpha_n e^{\lambda_n t} = 0 \implies \alpha_n = 0$ too. ∴ Linear independence

∎

Important: So if all roots of $P(\lambda)$ are distinct then the general solution of $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y^{(1)}(t) + a_0 y(t) = 0$ is

$$\boxed{y(t) = \sum_{j=1}^{n} \alpha_j e^{\lambda_j t} \qquad \alpha_j \in \mathbb{C}.}$$

Ex 1: $y'''(t) + 2y''(t) - 3y'(t) = 0.$

$$P(\lambda) = \lambda^3 + 2\lambda^2 - 3\lambda = 0$$
$$= \lambda(\lambda+3)(\lambda-1) = 0$$
$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = -3,\ \lambda_3 = 1.$$

General solution: $y(t) = C_1 + C_2 e^{-3t} + C_3 e^{t}.$

Ex 2: $y'''(t) - y'(t) = 0.$

$$P(\lambda) = \lambda^3 - \lambda = 0$$
$$= \lambda(\lambda-1)(\lambda+1) = 0$$
$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1.$$

ⓐ Initial condition: $y(0) = 1,\ y'(0) = 0,\ y''(0) = 3$

$$y(0) = 1 = C_1 + C_2 + C_3$$
$$y'(t) = -C_2 e^{-t} + C_3 e^{t} \rightarrow y'(0) = -C_2 + C_3 = 0$$
$$y''(t) = C_2 e^{-t} + C_3 e^{t} \rightarrow y''(0) = C_2 + C_3 = 3$$

$$\rightarrow C_3 = 3/2$$
$$C_2 = 3/2$$
$$C_1 = -2$$

$$\rightarrow \boxed{y(t) = -2 + \frac{3}{2} e^{-t} + \frac{3}{2} e^{t}}$$

⑥ Boundary Value Problem (BVP)

$$y(0) = 0, \quad y(1) = 2, \quad y'(0) = 0.$$

$$y(0) = 0 = C_1 + C_2 + C_3$$
$$y(1) = 0 = C_1 + C_2 e^{-1} + C_3 e^{1} = 2$$
$$y'(0) = 0 = -C_2 + C_3$$

$$\Rightarrow C_3 = C_2$$
$$C_2 = -1/2 \, C_1$$

$$C_1 = 2 - C_2 (e^{1} + e^{-1})$$

$$\rightarrow C_2 = \frac{2}{(e^{1}+e^{-1}) - 2} = \frac{1}{\cosh(1) - 1}$$

Lecture 5

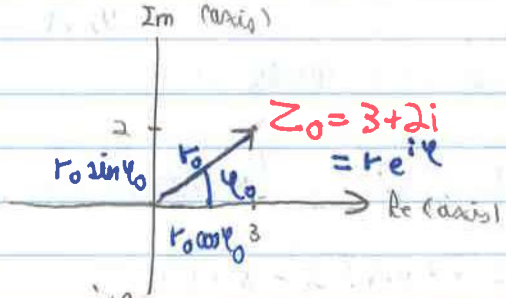
§ 2.1.4 Complex roots

$z = a + ib$, $\quad a, b \in \mathbb{R}$. Here, $i = \sqrt{-1}$, so that $i^2 = -1$.

Real part

Imaginary part

Complex plane :



$z_0 = 3 + 2i \quad (a = 3, b = 2)$

$r_0 = \sqrt{3^2 + 2^2} = 13$

$\varphi_0 = \arctan(2/3)$

$$z_0 = 3 + 2i = r_0 e^{i\varphi_0} = r \cos\varphi_0 + i r_0 \sin\varphi_0$$

Important result (Euler's formula) : $e^{i\varphi} = \cos\varphi + i\sin\varphi$.

($\varphi = \pi$ yields $e^{i\pi} = -1$ or $e^{i\pi} + 1 = 0$).

Properties to keep in mind :
$$e^{(x+iy)t} = e^{xt} \cdot e^{iyt}$$
$$= e^{xt}(\cos(yt) + i\sin(yt))$$
$$= e^{xt}\cos(yt) + ie^{xt}\sin(yt).$$

Ex (Harmonic oscillator) : $y''(t) + \omega^2 y(t) = 0$

Characteristic polynomial : $P(\lambda) = \lambda^2 + \omega^2 = 0$

Roots : $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$

$$\Rightarrow y(t) = C_1 e^{-i\omega t} + C_2 e^{i\omega t}$$

By Euler's formula : $y(t) = C_1(\cos(\omega t) - i\sin(\omega t)) + C_2(\cos(\omega t) + i\sin(\omega t))$

$$\Leftrightarrow y(t) = [C_1 + C_2]\cos(\omega t) + i[C_2 - C_1]\sin(\omega t)$$

Let $C_1 + C_2 = K_1$ $\Rightarrow$ $\boxed{y(t) = K_1\cos(\omega t) + K_2\sin(\omega t).}$

$i[C_2 - C_1] = K_2$

↳ General solution to the harmonic oscillator; this formula is more useful than the one with $e^{i\omega t}$ and $e^{-i\omega t}$.

(You can check that we can always do this.)

Ex: $y'''(t) + y(t) = 0$. The characteristic polynomial is $P(\lambda) = \lambda^3 + 1 = 0$

$$\Rightarrow P(\lambda) = (\lambda + 1)(\lambda^2 - \lambda + 1)$$

$\Rightarrow$ Roots are $\lambda_1 = -1$

$\lambda_2 = 1/2 + i\sqrt{3}/2$

$\lambda_3 = 1/2 - i\sqrt{3}/2$

$$\Rightarrow y(t) = C_1 e^{-t} + C_2 e^{t(\frac{1}{2}+i\frac{\sqrt{3}}{2})} + C_3 e^{t(\frac{1}{2}-i\frac{\sqrt{3}}{2})}$$
$$= C_1 e^{-t} + e^{t/2}\left[C_2 e^{\frac{i\sqrt{3}t}{2}} + C_3 e^{-\frac{i\sqrt{3}t}{2}}\right]$$
$$= C_1 e^{-t} + e^{t/2}\left[K_2 \cos\left(\frac{\sqrt{3}}{2}t\right) + K_3 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$\therefore y(t) = C_1 e^{-t} + K_2 e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + K_3 e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Remark: If the characteristic polynomial has real coefficients and $\lambda = a+ib$ is a root, then so is $\bar{\lambda} = a-ib$.

In the ODE we get terms like

$$C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}.$$

As in the last example, this can always be written like

$$e^{at}\left(K_1\cos(bt) + K_2\sin(bt)\right).$$

§ 2.1.5 Multiple roots

Ex: $y'''(t) = 0 \rightarrow p(\lambda) = \lambda^3$. Roots: $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Naively, the general solution would be $y(t) = C_1 e^{0t} + C_2 e^{0t} + C_3 e^{0t}$
$= C_1 + C_2 + C_3$, which are not 3 linearly independent solutions!

General solution: $C_1 + C_2 t + C_3 t^2$.

Theorem: Assume that λ_0 is a multiple zero of the characteristic polynomial of multiplicity m, meaning

$$(\lambda_0 - \lambda)^m \text{ is a factor of } P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

Then $\{e^{\lambda_0 t}, te^{\lambda_0 t}, \dots, t^{m-1}e^{\lambda_0 t}\}$ are linearly independent solutions of the ODE $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = 0$.

Proof : Left as an exercise.

Summary : Consider the n^{th}-order linear homogeneous ODE

$$(*) \quad y^{(n)}(t) + a_{n-1} y^{(n)}(t) + \dots + a_1 y^{(1)}(t) + a_0 y(t) = 0,$$ where the coefficients $\{a_0, a_1, \dots, a_{n-1}\}$ are all constants.

Then,

i) Any solution of (1) is $C^{\infty}(\mathbb{R})$. (Infinitely continuously differentiable on the real line.)

ii) Superposition principle: Any linear combination of two solutions of (*) is also a solution of (*).

iii) There exists n linearly independent solutions

$$S = \{y_1, y_2, \dots, y_n\}$$

such that any solution of (*) belongs to $\text{Span}(S)$.

iv) The solution space of (*) is a linear space (vector space) of dimension n!.

How do we find S? → Characteristic polynomial of (*):

$$P(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0.$$

Denote by $\{\lambda_j\}_{j=1}^{n}$, the roots of $P(\lambda)$.

Then,

i) If all roots are single and real, then S is given by

$$S = \{ e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t} \}.$$

ii) If a root λ_j is complex, that is, $\lambda_j = a_j + ib_j$ $(b_j \neq 0)$, we find two independent solutions:

$$e^{a_j t} \cos(b_j t) \quad + e^{a_j t} \sin(b_j t)$$

(If $\lambda_j = a_j + ib_j$ $(b_j \neq 0)$ is a root, then so is its complex conjugate $\bar{\lambda}_j = a_j - ib_j$.)

$$\mapsto C_1 e^{\lambda_j t} + C_2 e^{\bar{\lambda}_j t} = C_1 e^{a_j t + ib_j t} + C_2 e^{a_j t - ib_j t}$$

$$= e^{a_j t} \left(C_1 e^{ib_j t} + C_2 e^{-ib_j t} \right)$$

$$= e^{a_j t} \left[K_1 \cos(b_j t) + K_2 \sin(b_j t) \right].$$

iii) Multiple roots: Suppose that there is root λ_j of the characteristic polynomial $P(\lambda)$ of multiplicity m:

$\rightarrow$ $(\lambda_j - \lambda)^m$ is a factor of $P(\lambda)$, equivalently, this means

$$\left. \frac{d^k P(\lambda)}{d\lambda^k} \right|_{\lambda = \lambda_j} = 0 \quad , \; k = 0, 1, \dots, m-1$$

and $$\left. \frac{d^m P(\lambda)}{d\lambda^m} \right|_{\lambda = \lambda_j} \neq 0 .$$

Then $\{ e^{\lambda_j t}, t e^{\lambda_j t}, t^2 e^{\lambda_j t}, \dots, t^{m-1} e^{\lambda_j t} \}$ are m linearly independent solutions.

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Remark: We see that from i) – iii), we can always find n independent solutions (provided the roots of the char. poly. are available).

This means we have a complete characterization of the solution space

Root	Multiplicity	Independent solutions
$\lambda_j \in \mathbb{R}$	1	$e^{\lambda_j t}$
	$m > 1$	$e^{\lambda_j t}, te^{\lambda_j t}, \dots, t^{m-1}e^{\lambda_j t}$
$\lambda_j = a_j + ib_j$ $(b_j \neq 0)$	1	$e^{a_j t}\cos(b_j t), e^{a_j t}\sin(b_j t)$
	$m > 1$	$e^{a_j t}\cos(b_j t), te^{a_j t}\cos(b_j t), \dots, t^{m-1}e^{a_j t}\cos(b_j t)$, $e^{a_j t}\sin(b_j t), te^{a_j t}\sin(b_j t), \dots t^{m-1}e^{a_j t}\sin(b_j t)$.

Ex: $y^{(4)}(t) + 2y''(t) - 8y'(t) + 5y(t) = 0$

$$P(\lambda) = \lambda^4 + 2\lambda^2 - 8\lambda + 5 = 0 = (\lambda-1)^2(\lambda^2 - 2\lambda + 5)$$

4 roots: $\lambda = 1$ (double) $\rightarrow e^t, te^t$

$\lambda = -1 \pm 2i \rightarrow e^{-t}\cos(2t), e^{-t}\sin(2t)$

$$\Rightarrow y(t) = (C_1 + C_2 t)e^t + e^{-t}(C_3\cos(2t) + C_4\sin(2t)).$$

Ex: $y^{(4)}(t) + 2y''(t) + y(t) = 0$

$$P(\lambda) = \lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 = (\lambda - i)^2(\lambda + i)^2.$$

4 roots: Double complex roots $\lambda = i$, $\lambda = -i$

4 independent solutions: $\cos t, \sin t, t\cos t, t\sin t$

$$\Rightarrow y(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t.$$

* * *

§2.2. Linear ODEs with constant coefficients and inhomogeneous RHS

ODE: $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = \underbrace{F(t)}_{\neq 0}$

Warning ⚠ : The solution space is not a linear space!

How to find general solutions?

(Use g instead of y...)

Idea: Suppose there are two different solutions $y_1(t)$ and $y_2(t)$. Define $w(t) = y_1(t) - y_2(t)$. Note that

$$y_1^{(n)}(t) + a_{n-1}y_1^{(n-1)}(t) + \dots + a_1 y_1'(t) + a_0 y_1(t) = F(t)$$

and

$$y_2^{(n)}(t) + a_{n-1}y_2^{(n-1)}(t) + \dots + a_1 y_2'(t) + a_0 y_2(t) = F(t)$$

Subtract

$$\Rightarrow w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \dots + a_1 w'(t) + a_0 w(t) = 0.$$

↳ $w(t) := y_1(t) - y_2(t)$ solves the homogeneous part of the ODE

This means $w(t)$ is in the span of homogeneous solutions $\{Y_1, Y_2, \dots, Y_n\}$ of the homogeneous ODE.

$$\Rightarrow \boxed{y_1(t) = y_2(t) + \sum_{j=1}^{n} \alpha_j Y_j(t), \quad \alpha_j \in \mathbb{R}}$$

4 roots : Double complex roots $\lambda = i$, $\lambda = -i$

4 independent solutions : $\cos t, \sin t, t\cos t, t\sin t$

$$\Rightarrow y(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t.$$

§2.2. Linear ODEs with constant coefficients and inhomogeneous RHS

ODE : $y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = \underbrace{F(t)}_{\neq 0}$

Warning ⚠ : The solution space is not a linear space!

How to find general solutions ?

(Use g instead of y...)

Idea : Suppose there are two different solutions $y_1(t)$ and $y_2(t)$. Define $w(t) = y_1(t) - y_2(t)$. Note that

$$y_1^{(n)}(t) + a_{n-1} y_1^{(n-1)}(t) + \ldots + a_1 y_1'(t) + a_0 y_1(t) = F(t)$$

and

$$y_2^{(n)}(t) + a_{n-1} y_2^{(n-1)}(t) + \ldots + a_1 y_2'(t) + a_0 y_2(t) = F(t)$$

Subtract

$$\Rightarrow w^{(n)}(t) + a_{n-1} w^{(n-1)}(t) + \ldots + a_1 w'(t) + a_0 w(t) = 0.$$

↳ $w(t) := y_1(t) - y_2(t)$ solves the homogeneous part of the ODE

This means $w(t)$ is in the span of homogeneous solutions $\{Y_1, Y_2, \ldots, Y_n\}$ of the homogeneous ODE

$$\Rightarrow \boxed{y_1(t) = y_2(t) + \sum_{j=1}^{n} \alpha_j Y_j(t) \ , \ \alpha_j \in \mathbb{R}}$$

Theorem : Let y_p be a particular solution of the inhomogeneous ODE

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = F(t)$$

and let $\{Y_1, Y_2, \dots, Y_n\}$ be n independent solutions of the corresponding homogeneous ODE. Then, any solution of the inhomogeneous ODE is given by

$$y(t) = y_p(t) + \sum_{j=1}^{n} \alpha_j Y_j(t) \quad , \quad \alpha_j \in \mathbb{R}.$$

* * *

Ex 1 : $y''(t) - 2y(t) = 5t$.

i) $y_p(t) = -\frac{5}{2}t$

ii) $P(\lambda) = \lambda^2 - 2 \rightarrow$ Roots are $\sqrt{2}$ and $-\sqrt{2}$
$\rightarrow$ Solutions are $e^{\sqrt{2}t}$ and $e^{-\sqrt{2}t}$.

$$\Rightarrow y(t) = -\frac{5t}{2} + C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}.$$

Ex 2 : $y'''(t) + y''(t) + y'(t) + y(t) = e^t$

i) $y_p(t) = \frac{1}{4}e^t$

ii) $P(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1 = \lambda^2(\lambda+1) + (\lambda+1) = (\lambda^2+1)(\lambda+1)$

Roots : $\lambda_1 = -1 \Rightarrow e^{-t}$
$\lambda_2 = i, \ \lambda_3 = -i \Rightarrow \cos t, \sin t$

General solution : $y(t) = C_1 e^{-t} + C_2 \cos t + C_3 \sin t + \frac{1}{4}e^t.$

- Key concept to remember

> Solution of the inhomogeneous ODE
>
> = particular solution + solution of the homogeneous ODE.

- Question: How do we find particular solutions?
  - Numerically ... (§5)
  - Transformation techniques (Laplace transform - §2.5)
  - Variation of constants (§1.2.1 and §2.4)
  - Tables (for specific inhomogeneities (RHS of ODEs):

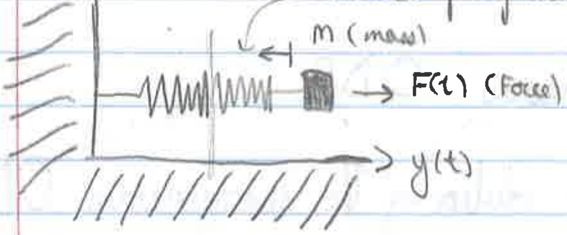
Table

$F(t)$	
$b_0+b_1t+b_2t^2+\dots+b_rt^r$ where $b_0, b_1, \dots, b_r \in \mathbb{R}$	$t^m\left(\sum_{k=0}^{r} A_k t^k\right)$, $r \in \mathbb{N}$, $m, A_k \in \mathbb{R}$
$e^{\alpha t}$	$C t^m e^{\alpha t}$ $C, m \in \mathbb{R}$
$\cos(\omega t)$ or $\sin(\omega t)$	$[A\cos(\omega t) + B\sin(\omega t)]\, t^m$ (often, $m = 0$ or 1)
$t^2 e^{-t}$	$(A_0 + A_1 t + A_2 t^2)\, e^{-t}$

Lecture 7

Ex: (Harmonic oscillator with external force)

Stretched spring with a mass attached on one end



By Newton's laws:

$$m y''(t) = F(t) - \underbrace{K y(t)}_{\text{Hooke's law}}$$

ODE: $y''(t) + \frac{k}{m} y(t) = F(t)$

In this example, let $\omega^2 = k/m$ and $F(t) = \cos(\omega t)$:

$$\to y''(t) + \omega^2 y(t) = \cos(\omega t)$$

i) Particular solution: Let's try $y_p(t) = t[A\cos(\omega t) + B\sin(\omega t)]$

Here, $y_p'(t) = [A\cos(\omega t) + B\sin(\omega t)] + t[-A\omega\sin(\omega t) + B\omega\cos(\omega t)]$

$$y_p''(t) = 2[-A\omega\sin(\omega t) + B\omega\cos(\omega t)] - t[A\omega^2\cos(\omega t) + B\omega^2\sin(\omega t)]$$

$$y_p''(t) + \omega^2 y_p(t) = 2[-A\omega\sin(\omega t) + B\omega\cos(\omega t)]$$

$$= \cos(\omega t) \iff A = 0 \text{ and } B = \frac{1}{2\omega}.$$

$$\therefore y_p(t) = \frac{t\sin(\omega t)}{2\omega}$$

ii) Homogeneous solution: $y''(t) + \omega^2 y(t) = 0$

Char. poly. $P(\lambda) = \lambda^2 + \omega^2 = (\lambda - i\omega)(\lambda + i\omega)$

Roots: $\lambda = i\omega$ and $\lambda = -i\omega$ $\Rightarrow$ Solutions: $\cos(\omega t)$, $\sin(\omega t)$.

, General solution

$$\Rightarrow y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{t}{2\omega}\sin(\omega t).$$

iii) Suppose we have the initial conditions: $y(0) = y'(0) = 0$, which corresponds to the mass m of the harmonic oscillator being at rest.

$$y(0) = C_1\cos(0) + C_2\sin(0) + \frac{0}{2\omega}\sin(0) = 0.$$

$$y'(0) = -\omega C_1\sin(0) + \omega C_2\cos(0) + \underline{\frac{\sin(0)}{2\omega}} + \underline{\frac{0\cdot\omega\cos(0)}{2\omega}}$$

$$= \omega C_2 = 0 \Rightarrow C_2 = 0$$

$\therefore$ The unique solution is $y(t) = \dfrac{t\sin(\omega t)}{2\omega}$.

__Note__: $\lim_{t\to\infty} |y(t)| = \infty$ (!) This behavior corresponds to harmonic resonance, a phenomenon that occurs when the frequency at which a force is periodically applied is equal or nearly equal to one of the natural frequencies of the system on which it acts.

(E.g., a swing at a playground, acoustic resonances of musical instrument...)

~~~~~

§2.3 Non-constant coefficients

This time, we will focus on the ODE

Problem ①+②

① $y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = F(t)$

with conditions

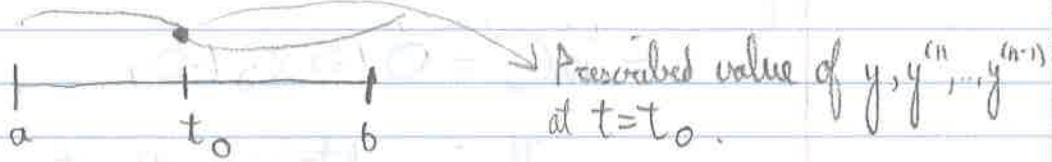
② $y(t_0) = y_0^{(0)}, y'(t_0) = y_0^{(1)}, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$.
~~~~~

We will also suppose that

a) The coefficients $a_0, a_1, \dots, a_{n-1}$, and the inhomogeneous term F are continuous on some open interval $I \subseteq \mathbb{R}$ (say, $I = (a,b)$ with $a < b$)

b) $y_0^{(0)}, y_0^{(1)}, \dots, y_0^{(n-1)} \in \mathbb{R}$ are given prescribed values.

c) The number $t_0 \in \bar{I}$ (the closure of the open interval I; if, say, $I = (a,b)$, then $\bar{I} = [a,b]$).



Theorem: Under these assumptions, problem ① + ② has exactly one solution on I. (That is, the solution is unique.)

Remark: If $F \not\equiv 0$, we can proceed in the same way as in the constant coefficients case:

$$\underset{\nearrow \atop \text{General solution}}{y(t)} = \underset{\nwarrow \atop \text{Particular solution}}{y_p(t)} + (\text{Solution for } F \equiv 0)$$

$\S 2.3.1$ Special case $F \equiv 0$

Theorem: There exists n independent solutions of (1) (with $F \equiv 0$):

$$y_1, \dots, y_n \quad (\text{fundamental set of solutions}).$$

Any solution of (1) can be written as a linear combination of the solutions to the fundamental set of solutions.

<u>Remark</u>: a) Any fundamental set of solutions is a basis of the linear solution space of (1).

b) The solution space of ① is n-dimensional.

- How does a solution of (1) + (2) look like? Consider a fundamental set of solutions of (1): $y_1, \dots, y_n$

$\Rightarrow$ The solution of (1)+(2) is given in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t).$$

What are the coefficients $c_1, c_2, \dots, c_n$? These are given by the initial conditions:

$$\begin{array}{rl} y(t_0) = & \boxed{\begin{array}{lcl} c_1 y_1(t_0) + \dots + c_n y_n(t_0) & = & y_0 \\ c_1 y_1'(t_0) + \dots + c_n y_n'(t_0) & = & y_0^{(1)} \\ & & \\ c_1 y_1^{(n)}(t_0) + \dots + c_n y_n^{(n)}(t_0) & = & y_0^{(n)} \end{array}} \\ y'(t_0) = & \\ \vdots & \\ y^{(n-1)}(t_0) = & \end{array}$$

↳ This is a linear system of n equations and n unknowns.

This can be written in matrix form (see next page!)

$$\underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0^{(1)} \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

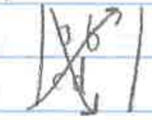
The determinant of this matrix is called the Wronskian (at t_0) of the ODE (1) (with $F \equiv 0$).

Facts: By uniqueness of the solution to (1)+(2), the coefficients $c_1, c_2, \dots, c_n$ are also unique!

As a consequence, the Wronskian is non-zero.

Aside: Determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \;;$$



We will denote the matrix as $M_{t_0}(y_1, \dots, y_n)$ and its determinant as $W_{t_0}(y_1, \dots, y_n)$

Proposition: The Wronskian $W_t(y_1, \dots, y_n) \neq 0$ for all $t \in I$ (eg. (a,b)) if and only if $y_1, \dots, y_n$ are linearly independent.

(See next page.)

Proof : "$\Rightarrow$" (only if) Suppose $M_t(y_1,\dots y_n)\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ for every $t\in I$.

(That is,

$$\begin{aligned}c_1y_1(t)+\dots+c_ny_n(t)&=0\\ c_1y_1'(t)+\dots+c_ny_n'(t)&=0\\ &\vdots\\ c_1y_1^{(n-1)}(t)+\dots+c_ny_n^{(n-1)}(t)&=0\end{aligned}$$

for every $t\in I$.)

Then as $W_t(y_1,\dots,y_n)\neq 0$ for every $t\in I$ by assumption, we must have

$$c_1=c_2=\dots=c_n=0$$

Therefore $y_1, y_2,\dots,y_n$ are linearly independent.

"$\Leftarrow$" (If) Suppose that $y_1,\dots,y_n$ are linearly independent and $W_T(y_1,\dots,y_n)=\underline{\underline{0}}$ for some $T\in I$.

Then $M_T(y_1,\dots,y_n)\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}=0$. By linear independence of $y_1,\dots,y_n$, there exists $\tilde{c}_1,\dots,\tilde{c}_n$ such that the general solution is

↑ for any choice of $\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$!

$y(t)=\tilde{C}_1y_1(t)+\dots+\tilde{C}_ny_n(t)$ and $\hat{c}_1,\dots,\hat{c}_n$ are not all zero.

By the fact that $M_T(y_1,\dots,y_n)\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}=0$ for any choice of $c_1,\dots,c_n$, we have that

$y(T)=\tilde{C}_1y_1(T)+\dots+\hat{C}_ny_n(T)=0$. Moreover, a similar argument gives

$y'(T)=\hat{c}_1y_1'(T)+\dots+\hat{C}_ny_n'(T)=0.$

$\vdots$

$y^{(n-1)}=\hat{C}_1y_1^{(n-1)}(T)+\dots+\tilde{C}_ny_n^{(n-1)}(T)=0$

$\Rightarrow y(T)=y'(T)=\dots=y^{(n-1)}(T)=0$

Taken together with uniqueness of solutions to (1)+(2), we find that $y(t) \equiv 0$ identically on I, that is,

$\Rightarrow \tilde{c}_1 y_1(t) + \dots + \tilde{c}_n y_n(t) = 0$ on I, with $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n$ not all zero.

$\Rightarrow y_1(t), \dots y_n(t)$ are linearly dependent. ∎

Lecture 8

Proposition : If $y_1, \dots, y_n$ are a fundamental set of solutions and $t \in I$, then

$$W_t(y_1, \dots, y_n) = e^{\left(\int_{t_0}^{t} -a_{n-1}(\tau)\,d\tau\right)} \cdot W_{t_0}(y_1, \dots, y_n)$$

↑ initial point.

This is known as the Abel-Jacobi-Liouville identity (or Liouville's formula).

Proof: We will do the proof for $n=2$.

$$W_t(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t) y_2'(t) - y_1'(t) y_2(t).$$

$$\frac{dW_t}{dt}(y_1, y_2) = y_1'(t) y_2'(t) + y_1(t) y_2''(t) - y_1'(t) y_2'(t) - y_1''(t) y_2(t)$$
$$= y_1(t) y_2''(t) - y_1''(t) y_2(t).$$

Now, replace $y_1''(t)$ and $y_2''(t)$ using the ODE:

$$y_1''(t) + a_1(t) y_1'(t) + a_0(t) y_1(t) = 0$$

$\Rightarrow y_1''(t) = -a_1(t) y_1'(t) - a_0(t) y_1(t)$ and similarly,

$y_2''(t) = -a_1(t) y_2'(t) - a_0(t) y_2(t)$. Now substitute in $\frac{dW_t}{dt}$:

$$\Rightarrow \frac{dW_t(y_1,y_2)}{dt} = -a_1(t)\,y_1(t)\,y_2'(t) - \cancel{a_0(t)\,y_1(t)\,y_2(t)}$$
$$+\, a_1(t)\,y_1'(t)\,y_2(t) + \cancel{a_0(t)\,y_1(t)\,y_2(t)}$$
$$= a_1(t)\left[-y_1(t)\,y_2'(t) + y_1'(t)\,y_2(t)\right] = -a_1(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$
$$= -a_1(t)\,W_t(y_1,\dots,y_n)$$

$\therefore \dfrac{dW_t(y_1,y_2)}{dt} = -a_1(t)\,W_t(y_1,y_2)$, and the solution can be found as in the separation of variable case:

$$W_t(y_1,y_2) = e^{\left(\int_{t_0}^{t} -a_1(s)\,ds\right)} \cdot C.$$ Set $t = t_0$ to find $C = W_{t_0}(y_1,y_2)$

$$\therefore W_t(y_1,y_2) = e^{\left(\int_{t_0}^{t} -a_1(s)\,ds\right)}\, W_{t_0}(y_1,y_2). \quad \blacksquare$$

Liouville's formula is useful because of the following corollary:

(Important) Corollary: Given n solutions $\{y_1, y_2, \dots, y_n\}$ of the ODE

① $$y^{n}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \dots + a_1(t)\,y'(t) + a_0(t)\,y(t) = F(t),$$

we have that $\{y_1, y_2, \dots, y_n\}$ are linearly independent if and only if the Wronskian $W_t(y_1,\dots,y_n)$ is non-zero for some $t \in I$ (see beginning of section §2.3.1.), and in particular if $W_{t_0}(y_1,\dots,y_n) \neq 0$.

→ This gives a mean to verify n solutions $\{y_1, y_2, \dots, y_n\}$ of the ODE (1) are linearly independent.

Ex: $2t^2 y''(t) + 3t y'(t) - y = 0, \quad t > 0$.

Then $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ form a fundamental set of solutions.

Indeed: $y_1'(t) = \frac{1}{2\sqrt{t}}$, $y_1''(t) = \frac{-1}{4t^{3/2}}$ and $y_2'(t) = -\frac{1}{t^2}$, $y_2''(t) = \frac{2}{t^3}$

ODE: $2t^2 \cdot \left(\frac{-1}{4t^{3/2}}\right) + 3t \cdot \frac{1}{2\sqrt{t}} - \sqrt{t} = -\frac{1}{2}\sqrt{t} + \frac{3}{2}\sqrt{t} - \sqrt{t} = 0$,

$2t^2 \cdot \left(\frac{2}{t^3}\right) + 3t \cdot \left(\frac{-1}{t^2}\right) - \frac{1}{t} = 4/t - 3/t - 1/t = 0$.

Are $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ linearly independent? Compute the Wronskian:

$$W_t(y_1, y_2) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

$$= \begin{vmatrix} \sqrt{t} & 1/t \\ \frac{1}{2\sqrt{t}} & -1/t^2 \end{vmatrix} = \sqrt{t} \cdot \left(\frac{-1}{t^2}\right) - \frac{1}{2\sqrt{t}} \cdot 1/t$$

$$= -\frac{3}{2t^{3/2}} \neq 0 \quad \forall t > 0.$$

∴ $\sqrt{t}$ and $1/t$ form a fundamental set of solutions, and the general solution to the ODE is $y(t) = C_1\sqrt{t} + C_2/t$.

Remark: How do we find a fundamental set to the ODE (1)?

↳ Series solutions, Laplace transforms, numerical solutions...

§2.4. Variation of constants

This is a general technique for handling inhomogeneous linear ODEs (with constant or non-constant coefficients $a_0, a_1, \ldots, a_{n-1}$) and is due by the Italian and French mathematician Joseph-Louis Lagrange (~1776).

Consider the n^{th}-order inhomogeneous ODE

$$(1)\quad \underset{\searrow\, \neq 0}{a_n(t)}\, y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = F(t).$$

Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the ODE (1).

<u>Idea</u>: Write the general solution of (1) as

$$y(t) = c_1(t)\,y_1(t) + \ldots + c_n(t)\,y_n(t)$$

Possibly non-constant coefficients!

(<u>Aside</u>: We did this already for first-order ODEs; see section §1.2.1 (ii).)

The idea now is to compute the derivatives of $y(t)$ up to order $n-1$, and along the way impose conditions on the coefficients $c_1(t), \ldots, c_n(t)$ and their first derivatives in way that will make $y(t)$ solve the ODE.

- $$y'(t) = \frac{d}{dt}\left(\sum_{k=1}^{n} c_k(t)y_k(t)\right) = \sum_{k=1}^{n}\left(c_k'(t)y_k(t) + c_k(t)y_k'(t)\right)$$

Impose $\sum_{k=1}^{n} c_k'(t)y_k(t) = 0$ so that $y'(t) = \sum_{k=1}^{n} c_k(t)y_k'(t)$

(We will see why soon.)

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- $y''(t) = \sum_{k=1}^{n} \left(c_k'(t) y_k'(t) + c_k(t) y_k''(t) \right)$

Impose $\sum_{k=1}^{n} c_k'(t) y_k'(t) = 0$.

⋮ Repeat up until the $(n-1)$ derivative,

$\Rightarrow$ Impose $\sum_{k=1}^{n} c_k'(t) y_k^{(j-1)}(t)$ for $j = 1, \dots, n-1$.

Also: $y^{(n)}(t) = \sum_{k=1}^{n} c_k'(t) y_k^{(n-1)}(t) + \sum_{k=1}^{n} c_k(t) y_k^{(n)}(t)$

Plug all of this in the ODE:

$$y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_1(t) y'(t) + a_0(t) y(t) = F(t)$$

$$\hookrightarrow \left(\sum_{k=1}^{n} c_k'(t) y_k^{(n-1)}(t) \right) + \sum_{k=1}^{n} c_k(t) \left[y_k^{(n)}(t) + a_{n-1}(t) y_k^{(n-1)}(t) + \dots + a_1(t) y_k'(t) + a_0(t) y_k(t) \right] = F(t).$$

But: $y_k^{(n)}(t) + a_{n-1}(t) y_k^{(n-1)}(t) + \dots + a_1(t) y_k'(t) + a_0(t) y_k(t) = 0$

for every $k = 1, \dots, n$ by assumption that $\{y_1, \dots, y_n\}$ is a fundamental set of solutions.

$\therefore$ We get $F(t) = a_n(t) \left[\sum_{k=1}^{n} c_k'(t) y_k^{(n-1)}(t) \right]$.

$\leftrightarrow \sum_{k=1}^{n} c_k'(t) y_k^{(n-1)}(t) = F(t) / a_n(t)$

$+ (n-1)$ conditions given by $\sum_{k=1}^{n} c_k'(t) y_k^{(j-1)}(t) = 0$, $j = 1, \dots, n-1$.

(See next page)

In matrix form:

$$\underbrace{\begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \\ y_1^{(n-1)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix}}_{n\times n} \begin{pmatrix} c_1'(t) \\ c_2'(t) \\ \vdots \\ \cdot \\ c_n^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F(t)/a_n(t) \end{pmatrix}$$

This is the matrix $M_t(y_1,\dots,y_n)$ that we encounted in the last section!

→ Its Wronskian, $W_t(y_1,\dots y_n) := \det(M_t(y_1,\dots,y_n))$ cannot be zero by assumption that $\{y_1,\dots,y_n\}$ is a fundamental set of solution.

<u>Procedure</u>: ① Solve for $c_1'(t),\dots c_n'(t)$

② Integrate: $c_k'(t) \longrightarrow c_k(t)$

③ Then $y(t) = \sum_{k=1}^{n} c_k(t)\, y_k(t)$.

<u>Ex:</u> $y''(t) - 5y'(t) + 6y = 2e^t$

① (Fundamental set of solutions) Compute the characteristic polynomial of the ODE:

$$P(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

↳ $y_1(t) = e^{3t}$, $y_2(t) = e^{2t}$.

② (Variation of constants): With $y_1'(t) = 3e^{3t}$ and $y_2'(t) = e^{2t}$:

6|/

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{pmatrix} 0 \\ 2e^{t} \end{pmatrix}$$

$$\Leftrightarrow \begin{matrix} \textcircled{\scriptsize I} \\ \textcircled{\scriptsize II} \end{matrix} \begin{bmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{pmatrix} 0 \\ 2e^{t} \end{pmatrix}$$

$$\textcircled{\scriptsize II} - 3\textcircled{\scriptsize I} \Leftrightarrow \begin{bmatrix} e^{3t} & e^{2t} \\ 0 & -e^{2t} \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} \begin{pmatrix} 0 \\ 2e^{t} \end{pmatrix}$$

$$\Rightarrow c_2'(t) = -2e^{-t} \Rightarrow c_2(t) = 2e^{-t} + K_2$$

$$\Rightarrow e^{3t} c_1'(t) + e^{2t} c_2'(t) = 0 \Leftrightarrow c_1'(t) + e^{-t}\left[-2e^{-t}\right] = 0$$

$$\Rightarrow c_1'(t) = 2e^{-2t}$$

$$\Rightarrow c_1(t) = -e^{-2t} + K_1$$

$$\therefore y(t) = (K_1 - e^{-2t})e^{3t} + (K_2 + 2e^{-t})e^{2t}.$$

$$= C_1 e^{3t} + C_2 e^{2t} + e^{t}$$

Lecture 9

§2.5 The Laplace transform

Idea:

$$\begin{tikzcd} \boxed{\text{ODE}} \arrow[r, "\text{Laplace Transform}"] & \text{Algebraic equation} \arrow[d] \\ \text{Solution.} & \text{Solve it! (Algebraically.)} \arrow[l, "\text{Invert the Laplace transform}"'] \end{tikzcd}$$

This method is often used in engineering and in the theory of Fourier transforms.

§2.5.1 Definition and examples

Defn: The Laplace transform takes a piecewise continuous function

$$f : \mathbb{R}_+ \to \mathbb{R} \ (\text{or } \mathbb{C}) \qquad (\mathbb{R}_+ = [0, \infty))$$

and maps it to a new function

$$\mathcal{L}[f](s) := F(s) = \int_0^\infty e^{-st} f(t)\,dt . \qquad \left(\begin{array}{l} \text{Notation:} \\ f(t) \circ\!\!-\!\!\bullet F(s) \end{array} \right)$$

Theorem: Suppose that f is a piecewise continuous function on $\mathbb{R}_+$ and that there exists constants $K, \gamma, M \in \mathbb{R}$ such that the following growth estimate is satisfied:

$$|f(t)| \leq K e^{\gamma t}, \quad \forall t \geq M.$$

Then $\mathcal{L}[f]$ is well-defined for $s > \gamma$.

Proof: $$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) = \underbrace{\int_0^M e^{-st} f(t)\,dt}_{\text{Finite.}} + \int_M^\infty e^{-st} f(t)\,dt$$

Now, we verify that $\left|\int_M^\infty e^{-st} f(t)\,dt\right|$ is finite:

$$\left|\int_M^\infty e^{-st} f(t)\,dt\right| \le \int_M^\infty e^{-st} \underbrace{|f(t)|}_{\le Ke^{at}}\,dt$$

$$\le \int_M^\infty e^{-st} Ke^{at}\,dt = K\int_M^\infty e^{-(s-a)t}\,dt$$

$< \infty$ provided $s-a>0$. ■

Examples:

① $f(t)=\cos(\omega t)$ (growth assumption is satisfied trivially with $M=0$; $|\cos(t)| \le 1$).

Then $F(s) = \int_0^\infty e^{-st}\cos(\omega t)\,dt$. Integrate by parts:

$u = \cos(\omega t)$ $\quad dv = e^{-st}\,dt$

$du = -\omega\sin(\omega t)$ $\quad v = -\frac{1}{s}e^{-st}$

$$= uv - \int v\,du$$

$$= \underbrace{-\frac{\cos(\omega t)}{s} e^{-st}\Big|_0^\infty}_{=1/s} - \frac{\omega}{s}\int e^{-st}\sin(\omega t)\,dt$$

$$= 1/s - \omega/s\int_0^\infty e^{-st}\sin(\omega t)$$

$u = \sin(\omega t)$ $\quad dv = e^{-st}\,dt$

$du = \omega\cos(\omega t)$ $\quad v = -\frac{1}{s}e^{-st}$

$$= 1/s + \underbrace{\frac{\omega\sin(\omega t)}{s} e^{-st}\Big|_0^\infty}_{=0} - \frac{\omega}{s}\cdot\underbrace{\int_0^\infty \frac{\omega}{s} e^{-st}\cos(\omega t)}_{F(s)}$$

$$\therefore F(s) = \frac{1}{s} - \frac{\omega^2}{s^2} F(s) \Rightarrow F(s) = \frac{1}{s(1+\frac{\omega^2}{s^2})} = \frac{s}{(s^2+\omega^2)}$$

$$\therefore \boxed{\cos(\omega t) \circ\!\!-\!\!\!-\!\!\bullet \frac{s}{s^2+\omega^2}}$$

② $f(t) = 1$.

Then $F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$. So $\boxed{1 \circ\!\!-\!\!\!-\!\!\bullet \, {}^1\!/_s}$

③ $f_n(t) = t^n$ and let $F_n(s) = \mathcal{L}[t^n](s)$.

$$\text{Then } F_n(s) = \int_0^\infty e^{-st} t^n\,dt \overset{\text{I.B.P.}}{=} \underbrace{-\frac{t^n e^{-st}}{s}\Big|_0^\infty}_{=0} + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt.$$

$$\hookrightarrow F_n(s) = \frac{n}{s} F_{n-1}(s) = \frac{n(n-1)}{s^2} F_{n-2}(s)$$

$$= \frac{n!}{s^n} F_0(s) = \frac{n!}{s^n} \mathcal{L}[1](s) = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$\therefore \boxed{t^n \circ\!\!-\!\!\!-\!\!\bullet \frac{n!}{s^{n+1}}}$$

④ $f(t) = e^{\lambda t}$: $F(s) = \int_0^\infty e^{-st} e^{\lambda t}\,dt = \frac{1}{\lambda - s} e^{(\lambda - s)t}\Big|_0^\infty$

$= \frac{1}{s-\lambda}$ provided $s - \lambda > 0$ $(s > \lambda)$

$\therefore e^{\lambda t} \circ\!\!-\!\!\!-\!\!\bullet \, {}^1\!/_{s-\lambda}$ provided $s > \lambda$.

§2.5.2 Rules

Theorem : With $f \circ\!\!-\!\!\bullet F$, then there holds:

a) $\begin{cases} 0, & t<\lambda \\ f(t-\lambda), & t\geq 0 \end{cases} \circ\!\!-\!\!\bullet \; e^{-s\lambda}F(s)$ $(\lambda>0)$.

b) $f(\alpha t) \circ\!\!-\!\!\bullet \frac{1}{\alpha}F(s/\alpha)$ $(\alpha>0)$

c) $e^{\lambda}f(t) \circ\!\!-\!\!\bullet F(s-\lambda)$ $(\lambda\in\mathbb{C})$

d) $t^n f(t) \circ\!\!-\!\!\bullet (-1)^n F^{(n)}(s)$ ← n^{th} derivative.

e) $\frac{f(t)}{t} \circ\!\!-\!\!\bullet \int_s^\infty F(\sigma)d\sigma$

f) $f'(t) \circ\!\!-\!\!\bullet sF(s)-f(0^+)$ $(0^+ = \lim_{x\to 0^+} x$, limit from the right).

g) $\int_0^t f(\sigma)d\sigma \circ\!\!-\!\!\bullet \frac{F(s)}{s}$.

h) $\mathcal{L}[\alpha f+\beta g](s) = \alpha\mathcal{L}[f](s)+\beta\mathcal{L}[f](s)$, α and $\beta \in \mathbb{R}$ (or $\mathbb{C}$).

Proof: (a) and f) - the rest are left as an exercise).

a) : $\begin{cases} 0 & t<\lambda \\ f(t-\sigma) & t\geq\lambda \end{cases} \circ\!\!-\!\!\bullet \; e^{-s\lambda}F(s)$

$$\int_\lambda^\infty e^{-st}f(t-\lambda)\,dt \overset{\sigma=t-\lambda}{=} \int_0^\infty e^{-s(\sigma+\lambda)}f(\sigma)\,d\sigma = e^{-s\lambda}F(s).$$

f) : $f'(t) \circ\!\!-\!\!\!-\!\!\bullet\ sF(s) - f(0^+)$

$$\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t)\,dt \qquad \begin{array}{ll} u = e^{-st} & dv = f'(t)\,dt \\ du = -se^{-st}\,dt & v = f(t) \end{array}$$

$$\overset{\text{I.B.P.}}{=} e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0^+) + sF(s) \quad \left(\text{growth condition ensures } \lim_{t\to\infty} e^{-st} f(t) = 0.\right)$$

Rest: Exercise ... ■

Note: We can generalize f) :

$$\mathcal{L}[f''](s) = -f'(0^+) + s\,\mathcal{L}[f'](s) = -f'(0^+) - sf(0^+) + s^2 F(s).$$

In general :

$$\boxed{\mathcal{L}[f^{(n)}](s) = s^n F(s) - f^{(n-1)}(0^+) - sf^{(n-2)}(0^+) - \dots - s^{n-1} f(0^+).}$$

§ 2.5.3. Laplace transform and ODEs

Idea :

$$\begin{aligned} y(t) &\circ\!\!-\!\!\!-\!\!\bullet\ Y(s) \\ y'(t) &\circ\!\!-\!\!\!-\!\!\bullet\ -y(0^+) + sY(s) \\ y''(t) &\circ\!\!-\!\!\!-\!\!\bullet\ -y'(0^+) - sy(0^+) + s^2 Y(s) \end{aligned}$$

E.g 1: Consider the ODE $y''(t) - y'(t) - 2y(t) = 0$ with $y(0)=1$, $y'(0)=0$

$$\Rightarrow \mathcal{L}[y''-y'-2y](s) = \Big(-\underbrace{y'(0^+)}_{=0} - s\underbrace{y(0^+)}_{=1} + s^2 Y(s)\Big) - \Big(-\underbrace{y(0^+)}_{=1} + sY(s)\Big) - 2\big(Y(s)\big)$$

$$= -s + s^2 Y(s) + 1 - sY(s) - 2Y(s) = \mathcal{L}[0](s) = 0.$$

$$Y(s)\underbrace{\left[s^2 - s - 2\right]}_{=(s-2)(s+1)} = s-1$$

$$\Rightarrow Y(s) = \frac{(s-1)}{(s-2)(s+1)} \overset{\text{Partial Fraction}}{=} \frac{A}{(s-2)} + \frac{B}{(s+1)}, \text{ where } \begin{matrix} A+B=1 \\ A-2B=-1 \end{matrix}$$

We have $2A - B = 0$ or $A = B/2$ and $B = 2/3 \rightarrow A = 1/3$

So $Y(s) = \frac{1}{3} \cdot \frac{1}{(s-2)} + \frac{2}{3}\frac{1}{(s+1)}$

But $\frac{1}{3}\frac{1}{s-2} \bullet\!\!-\!\!\circ \frac{1}{3}e^{2t}$ and $\frac{2}{3}\frac{1}{(s+1)} \bullet\!\!-\!\!\circ \frac{2}{3}e^{-t}$

$$\therefore Y(s) \bullet\!\!-\!\!\circ \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Ex 2: $y'(t) + 3y(t) = \cos t$, $y(0)=0$

$$\begin{array}{c}\circ \\ | \\ \bullet\end{array}$$

$$-\underbrace{y(0^+)}_{=0} + sY(s) + 3Y(s) = \frac{s}{s^2+1} \Rightarrow Y(s) = \frac{s}{(s+3)(s^2+1)}$$

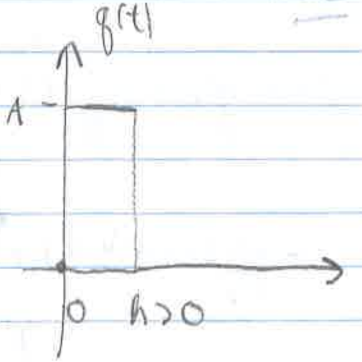
$$\overset{\text{Partial fractions}}{=} \frac{\frac{1}{10}(3s+1)}{s^2+1} - \frac{3}{10}\cdot\frac{1}{s+3}$$

$$= \frac{1}{10}\left[\frac{3s}{s^2+1} + \frac{1}{s^2+1} - \frac{3}{s+3} \right] = Y(s)$$

$$\bullet\!\!-\!\!\circ$$

$$\frac{1}{10}\left[3\cos t + \sin t - 3e^{-3t} \right] = y(t).$$

§2.5.4 Discontinuous functions



Ex: Consider an impulse of magnitude A:

$$g(t) = \begin{cases} A, & 0 \leq t \leq h \\ 0, & t > h \end{cases}.$$

What is its Laplace transform?

$$\mathcal{L}[g](s) = \int_0^\infty e^{-st} g(t)\,dt = \int_0^h Ae^{-st}\,dt = -\frac{A}{s}e^{-st}\Big|_0^h$$

$$= \frac{A}{s}\left[1 - e^{-sh}\right]$$

Defn: The function $t \longmapsto H(t) := \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$ is called the Heaviside function. Its Laplace transform is $\mathcal{L}[H](s) = 1/s$.

In general: Express step functions (piecewise constants) in terms of Heaviside functions

$$\mathcal{L}[H(\cdot - h)] = \frac{1}{s}e^{-sh}.$$

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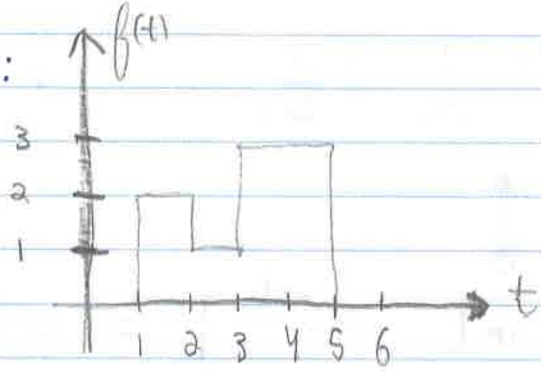
Ex:



Then $f(t) = 2H(t-1) - H(t-2) + 2H(t-3)$
$-3H(t-5).$

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$F(s) = \frac{2e^{-s}}{s} - \frac{1e^{-2s}}{s} + \frac{2e^{-3s}}{s} - \frac{3e^{5s}}{s}.$$

Ex: ODE: $y'(t) + 3y(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$, $y(0) = 0.$

$\Leftrightarrow$ $y'(t) + 3y(t) = H(t) - H(t-1)$

$$\begin{array}{c}\circ\\ |\\ \bullet\end{array}$$

$$-\underset{=0}{y(0^+)} + sY(s) + 3Y(s) = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$\Updownarrow$$

$$Y(s) = \frac{1}{s(s+3)}\left[1 - e^{-s}\right].$$ let $F(s) = \frac{1}{s(s+3)}.$

Then $Y(s) = F(s) - F(s)e^{-s}$

$$\begin{array}{cc}\circ & \bullet\\ | & |\\ \circ & \circ\end{array}$$

$$f(t), \begin{cases} f(t-1), & t \geq 1 \\ 0, & t < 1 \end{cases}$$

Hence $y(t) = \begin{cases} f(t) & t < 1 \\ f(t) - f(t-1), & t \geq 1 \end{cases}$

Now, $F(s) = \frac{1}{s(s+3)} \overset{\text{Partial Fractions}}{=} \frac{1}{3}\left[\frac{1}{s} - \frac{1}{(s+3)}\right]$

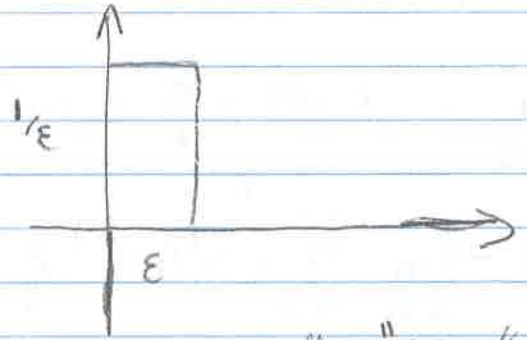
$\Rightarrow$ $F(s) \bullet\!\!-\!\!\circ \frac{1}{3}\left[1 - e^{-3t}\right] = f(t)$

$$\Rightarrow y(t) = \begin{cases} \frac{1}{3}(1 - e^{-3t}), & t < 1 \\ \frac{1}{3}(e^{-3(t-1)} - e^{-3t}), & t \geq 1. \end{cases}$$

Lecture 10

Last time: Heaviside function $H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

Special case: The Dirac δ function "δ-function".



$$\delta_\varepsilon(t) = \frac{1}{\varepsilon} H(t) - \frac{H(t-\varepsilon)}{\varepsilon}$$

We then "define" $\delta_0 = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$ (distribution or generalized function)

The Dirac δ-function δ_0 gives "unit mass" at $t = 0$ in the following sense: For a function $f : \mathbb{R}_+ \to \mathbb{R}$, there holds

$$\int_0^\infty f(x)\, \delta_0(x)\, dx \overset{\text{"\,"} \to \text{formally}}{=} \lim_{\varepsilon \to 0} \int_0^\infty f(x)\, \delta_\varepsilon(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \underbrace{\frac{1}{\varepsilon} \int_0^\varepsilon f(x)\, dx}_{\text{Average over an interval } (0,\varepsilon).} \;\text{"}{=}\text{"}\; f(0).$$

[illegible]

Formally, the Laplace transform of the Dirac δ-function is

$$\mathcal{L}[\delta_0](s) \;\text{"}{=}\text{"}\; \lim_{\varepsilon \to 0} \mathcal{L}[\delta_\varepsilon] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\mathcal{L}[H(t)] - \mathcal{L}[H(t-\varepsilon)] \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{1}{s} - e^{-\varepsilon s} \cdot \frac{1}{s} \right) = \lim_{\varepsilon \to 0} \frac{1 - e^{-\varepsilon s}}{s\varepsilon}.$$

Note: $e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = 1 + x + \frac{x^{2}}{2} + \ldots$

So $e^{-\varepsilon s} = 1 - \varepsilon s + \sum_{k=2}^{\infty} \frac{(-\varepsilon s)^{k}}{k!}$

$$\text{So } \lim_{\varepsilon \to 0} \frac{1 - e^{-\varepsilon s}}{\varepsilon s} = \lim_{\varepsilon \to 0} \frac{\left(1 - 1 + \varepsilon s - \sum_{k=2}^{\infty} (-\varepsilon s)^{k}/k!\right)}{\varepsilon s}$$

$$= 1 - \underbrace{\lim_{\varepsilon \to 0} \sum_{k=2}^{\infty} (-1)^{k} (\varepsilon s)^{k-1}/k!}_{=0}$$

$$\therefore \mathcal{L}[\delta_0] = 1, \quad \delta_0(t) \circ\!\!-\!\!\!\bullet\ 1.$$

We will see why this is useful in section §2.5.6 on Green's functions.

§2.5.5 The convolution rule

Defn: For two functions f and g, the convolution $f * g$ is given by:

$$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= (g*f)(t)$$

Theorem: Suppose that f and g satisfy the growth estimates

$$|f(t)| \le Ke^{\alpha t}, \quad |g(t)| \le Ke^{\alpha t},$$

where $K > 0$, $\alpha \in \mathbb{R}$, $t > 0$. Then $(f*g)$ is well defined ($|(f*g)(t)| < \infty$) and

$$(f*g)(t) \circ\!\!-\!\!\!\bullet\ F(s)\,G(s), \quad \text{where } F(s) = \mathcal{L}[f](s) \text{ and } G(s) = \mathcal{L}[g](s)$$

Step 1 **Proof:** $|(f*g)(t)| = \left|\int_0^t f(t-\tau)g(\tau)d\tau\right| \le \int_0^t |f(t-\tau)||g(\tau)|d\tau$

Growth estimate $\le \int_0^t Ke^{\alpha(t-\tau)}Ke^{\alpha\tau}d\tau \le K^2e^{\alpha t}t$

Since $1+t \le e^t$ for every $t\in\mathbb{R}$, we find

$$|(f*g)(t)| \le K^2e^{\alpha t}t \le K^2e^{(\alpha+1)t} = \tilde{K}e^{\tilde{\alpha}t}, \quad \tilde{K}=K^2, \ \tilde{\alpha}=\alpha+1$$

$\Longrightarrow$ $f*g$ is well-defined and $\mathcal{L}[f*g]$ is also well-defined.

Step 2 Let's compute the Laplace transform $\mathcal{L}[f*g](s)$:

$$\mathcal{L}[f*g](s) = \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right)dt$$

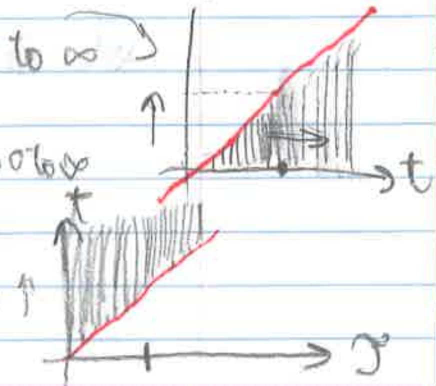
$$= \int_0^\infty\left(\int_0^t e^{-st}f(t-\tau)g(\tau)d\tau\right)dt$$

$$= \int_0^\infty\left(\int_0^t(e^{-s\tau}g(\tau))(e^{-s(t-\tau)}f(t-\tau))d\tau\right)dt$$

We can interchange the order of integrals (proof omitted).

↳ Integrate τ from 0 to t and t from 0 to ∞

is the same as integrating t from τ to ∞ and τ from 0 to ∞



$$\Rightarrow (f*g)(t) = \int_0^\infty \int_\sigma^\infty \left(e^{-s\sigma} g(\sigma)\right)\left(e^{-s(t-\sigma)} f(t-\sigma)\right) dt\, d\sigma$$

$$= \int_0^\infty \left(e^{-s\sigma} g(\sigma)\right) \int_\sigma^\infty e^{-(s)(t-\sigma)} f(t-\sigma)\, dt\, d\sigma$$

Let $T = t-\sigma$

$dT = dt$

$$= \int_0^\infty e^{-s\sigma} g(\sigma) \int_0^\infty e^{-sT} f(T)\, d\sigma\, dT$$

$$= \underbrace{\left(\int_0^\infty e^{-s\sigma} g(\sigma)\, d\sigma\right)}_{G(s)} \underbrace{\left(\int_0^\infty e^{-sT} f(T)\, dT\right)}_{F(s)}$$

Ex: $f(t) = \sqrt{t}$. What is its Laplace transform?

$$(f*f)(t) = \int_0^t \sqrt{\sigma(t-\sigma)}\, d\sigma$$. Let $\sigma = \frac{t}{2} + x$. $d\sigma = dx$

$-\frac{t}{2} \leq x \leq t/2$

$$= \int_{-\frac{t}{2}}^{t/2} \sqrt{\left(\tfrac{t}{2}+x\right)\left(\tfrac{t}{2}-x\right)} = \int_{-t/2}^{t/2} \sqrt{\tfrac{t^2}{4} - x^2}$$

$$\overset{\text{Trigonometric substitution}}{=} \pi t^2/8$$

But $(f*f)(t) \circ\!\!-\!\!\bullet\ F(s)^2$

$$\begin{array}{ccc} (f*f)(t) & \circ\!\!-\!\!\bullet & F(s)^2 \\ \| & & \| \\ \frac{\pi t^2}{8} & & \frac{\pi}{8}\mathcal{L}[t^2] = \frac{\pi}{8}\frac{2!}{s^3} \end{array}$$

$$\therefore \sqrt{t} \circ\!\!-\!\!\bullet \frac{\sqrt{\pi}}{2s^{3/2}}.$$

§ 2.5.6 Green's functions

Ex: $y'(t) + ay = f(t)$, $y(0) = 0$

$$\begin{array}{c}\circ\\ |\\ \bullet\end{array}$$

$$sY(s) - y(0^-) + aY(s) = F(s) \Rightarrow Y(s) = \underbrace{\left(\frac{1}{(s+a)}\right)}_{:= G(s)} F(s).$$

Depends on the homogeneous part of the ODE → $\frac{1}{(s+a)}$

Depends of $f(t)$ ↑ $F(s)$

We call $G(s)$ the transfer function of the (homogeneous part of the) ODE.

In general: $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \dots + a_1 y'(t) + a_0 y(t) = f(t)$, where $\{a_0, a_1, \dots, a_{n-1}\}$ are constant coefficients, and $y(0) = y'(0) = \dots = y^{(n-1)}(0)$.

$$\Rightarrow Y(s) = \frac{1}{P(s)} \cdot F(s)$$

Characteristic polynomial → $P(s)$

$F(s)$ ↘ $\mathcal{L}[f](s)$

↘ $\frac{1}{P(s)} = G(s)$ is the transfer function.

Special case: Suppose $f(t)$ is the Dirac δ-function $\delta_0(t)$. Then $\mathcal{L}[\delta_0](s) = 1$, and in that case,

$$Y(s) = \frac{1}{p(s)} = G(s)$$

Defn: We call the inverse Laplace transform of $G(s)$, $\mathcal{L}^{-1}[G](t)$, the Green function for the given ODE (that doesn't depend on the Right-hand side of the ODE).

Back to the general case: $Y(s) = G(s) \cdot F(s)$ (convolution!)

We find $Y(s) = G(s)F(s)$

$$y(t) = \underset{\nwarrow \text{ inverse laplace transform}}{\mathcal{L}^{-1}}(G\cdot F)(t) = (g*f)(t)$$

We can solve the ODE (find $y(t)$) using the following procedure :

Procedure : ① Compute the Green's function $g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t)$, where $P(s)$ is the characteristic polynomial of the ODE.

② Solve the ODE : $y(t) = (g*f)(t)$ for any inhomogeneous term f.

$\Rightarrow$ Compute the Green's function once for a given ODE, and any solution can be found using ②.

Ex: (Harmonic oscillator)

$$y''(t) + \omega^2 y(t) = f(t) \quad , \quad y(0) = y'(0) = 0.$$

① Find the Green's function : The characteristic polynomial of $y''(t) + \omega^2 y(t)$ is $p(s) = s^2 + \omega^2$ and $\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{\sin(\omega t)}{\omega} \equiv g(t)$

② $$y(t) = (g*f)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

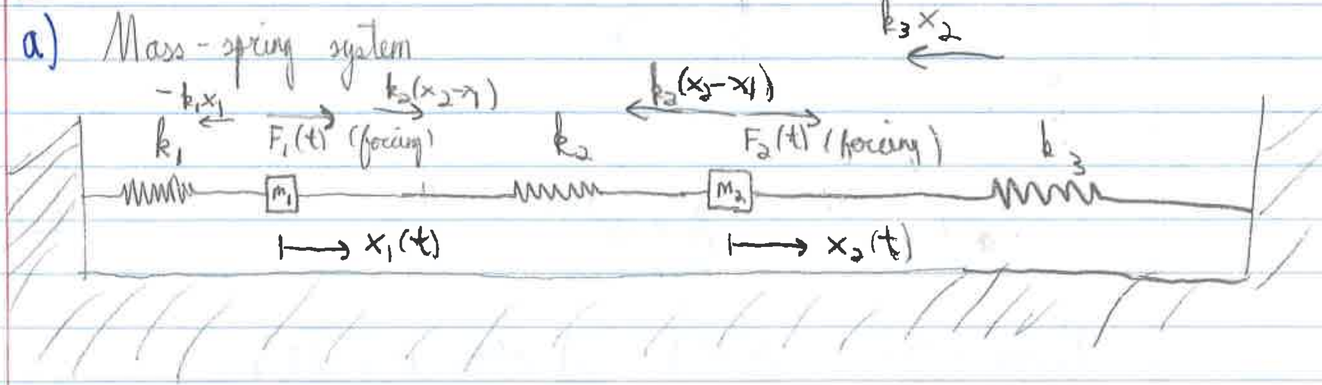
$$= \frac{1}{\omega}\int_0^t f(\tau)\sin(\omega(t-\tau))\, d\tau \, . \quad t>0$$

END OF MATERIAL FOR THE MIDTERM

§3 Linear systems of ODEs

§3.1 Examples

a) Mass-spring system



$$m_1 x_1''(t) = F_1(t) - k_1 x_1(t) + k_2(x_2(t) - x_1(t))$$

$$m_2 x_2''(t) = F_2(t) - k_2(x_2(t) - x_1(t)) - k_3 x_2(t)$$

Let $y_1(t) = x_1'(t)$ and $y_2(t) = x_2'(t)$. Then

$y_1'(t) = x_1''(t)$ and $y_2'(t) = x_2''(t)$ and so we find

$$x_1'(t) = y_1(t)$$
$$x_2'(t) = y_2(t)$$

$$y_1'(t) = \frac{1}{m_1}\left[F_1(t) - (k_1+k_2)x_1(t) + k_2 x_2(t)\right]$$

$$y_2'(t) = \frac{1}{m_2}\left[F_2(t) - (k_2+k_3)x_2(t) + k_2 x_1(t)\right]$$

Write all of this in matrix-vector form: $\underline{z}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow \underline{z}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ y_1'(t) \\ y_2'(t) \end{pmatrix}$

$$\Rightarrow \underline{z}'(t) = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1+k_2}{m_1}\right) & k_2/m_1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & 0 & 0 \end{pmatrix}}_{\underline{\underline{A}} \in \mathbb{R}^{4\times 4}} \underline{z}(t) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ F_1(t)/m_1 \\ F_2(t)/m_2 \end{pmatrix}}_{\underline{F}(t)}$$

Vector ODE: $\underline{z}'(t) = \underline{\underline{A}}\, z(t) + \underline{F}(t)$

(Notation: I use one underline to denote a vector $\underline{v}$ and two underlines to denote a matrix $\underline{\underline{B}}$.)

b) n^{th}-order ODEs. Suppose we have the inhomogeneous ODE

$$y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) = f(t)$$

Change of variable: let $z_1(t) = y(t)$, $z_2(t) = y'(t)$, ... $z_n(t) = y^{(n-1)}(t)$

$$\hookrightarrow z_1'(t) = y'(t) = z_2(t)$$

$$z_2'(t) = y''(t) = z_3(t)$$

$$\vdots$$

$$z_n'(t) = y^{(n)}(t) = f(t) - [a_{n-1}(t) z_n(t) + a_{n-2}(t) z_{n-1}(t) + a_1(t) z_2(t) + a_0(t) z_1(t)]$$

$$\Rightarrow \underline{z}'(t) = \underline{\underline{A}}(t)\, \underline{z}(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix} \quad \text{where } \underline{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}$$

↗ Matrix of coefficients

Lecture 11

§3.2 Linear algebra review II

$\underline{\underline{A}}, \underline{\underline{B}} \in \mathbb{R}^{m \times n}$, $\underline{v} \in \mathbb{R}^n$, and suppose $(\underline{\underline{A}})_{ij} = a_{ij}$ and $(\underline{v})_j = v_j$
$i = 1, \ldots, m$ $\quad$ $j = 1, \ldots, n$

↳ Then $(\underline{\underline{A}}\,\underline{v})_i = \sum_{k=1}^{n} a_{ik} v_k$.

a) $+, -$, scalar multiple component-wise : $(\underline{\underline{A}})_{ij} = a_{ij}$, $(\underline{\underline{B}})_{ij} = b_{ij}$

$$= \alpha(\underline{\underline{A}})_{ij} + \beta(\underline{\underline{B}})_{ij} = \alpha a_{ij} + \beta_{ij}$$

b) $*$, $m = n$: $(\underline{\underline{A}}\,\underline{\underline{B}})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

c) Eigenvalues and eigenvectors   Suppose $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ (given).

Find $\lambda \in \mathbb{C}$ (eigenvalues) and vectors $v \neq 0$ (eigenvectors) $\in \mathbb{R}^n$ such that

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

$$\Longleftrightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0} \overset{\underline{v} \neq 0}{\Longleftrightarrow} \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \neq 0$$

↳ $:= P(\lambda)$ (Characteristic polynomial)

Ex : $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \Rightarrow \underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Longleftrightarrow (\lambda - 2)(\lambda - 3) \Rightarrow \lambda_1 = 2, \ \lambda_2 = 3$$

What is the eigenvector for $\lambda_1 = 2$?

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v} \Longleftrightarrow \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \underline{0} \quad \Leftrightarrow \quad \underline{v} = \alpha\begin{pmatrix}2\\ 1\end{pmatrix} \quad \forall \alpha \in \mathbb{R}\setminus\{0\}$$

Setting $\alpha = 1$ yields $\underline{v} = \begin{pmatrix}2\\ 1\end{pmatrix}$, for example.

Similarly, the eigenvector for the eigenvalue $\lambda_2 = 3$ is $\beta\begin{pmatrix}1\\ 1\end{pmatrix}$, $\beta \in \mathbb{R}\setminus\{0\}$

$\hookrightarrow \beta_1 = 1 \Rightarrow \underline{v} = \begin{pmatrix}1\\ 1\end{pmatrix}$.

d) Diagonalization : If we can find n linearly independent eigenvectors for an $n \times n$ matrix (this is true, for example, if the roots of the char. poly $P(\lambda)$, i.e., the eigenvalues, are single roots), then the matrix can be diagonalized.

Define : $$\underline{\underline{T}} = \underbrace{\left[\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_n\right]}_{\text{Matrix whose columns are the eigenvectors of } \underline{A}} \Rightarrow \underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}$$

Ex (continued) : $A = \begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix} \Rightarrow \lambda_1 = 2$ with $\underline{v}_{\lambda_1} = \begin{pmatrix}2\\ 1\end{pmatrix}$

$\lambda_2 = 3$ with $\underline{v}_{\lambda_2} = \begin{pmatrix}1\\ 1\end{pmatrix}$

Then $\underline{\underline{T}} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$. We can calculate T^{-1} by inverting

$$\begin{matrix}\text{(I)}\\ \text{(II)}\end{matrix}\left(\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{array}\right) \xrightarrow{\text{(I)}-\text{(II)}} \left(\begin{array}{cc|cc}1 & 0 & 1 & -1\\ 1 & 1 & 0 & 1\end{array}\right)$$

$$\xrightarrow{\text{(II)}-\text{(I)}} \left(\begin{array}{cc|cc}1 & 0 & 1 & -1\\ 0 & 1 & -1 & 2\end{array}\right) \Rightarrow \underline{\underline{T}}^{-1} = \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

Check: $\underline{\underline{T}}\,\underline{\underline{T}}^{-1} = \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\-1&2\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ ✓

Then $\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \underline{\underline{T}}^{-1}\begin{pmatrix}1&2\\-1&4\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix} = T^{-1}\begin{pmatrix}4&3\\2&3\end{pmatrix} = \begin{pmatrix}1&-1\\-1&2\end{pmatrix}\begin{pmatrix}4&3\\2&3\end{pmatrix}$

$$= \begin{pmatrix}2&0\\0&3\end{pmatrix}$$ ✓

§ 3.3. First-order systems of ODEs

$$\underbrace{y'(t)}_{\in\mathbb{R}^n} = \underbrace{\underline{A}(t)}_{\mathbb{R}^{n\times n}}\,\underbrace{y(t)}_{\mathbb{R}^n} + \underbrace{\underline{F}(t)}_{\mathbb{R}^n} \qquad (\underline{1})$$

Theorem: Suppose that $\underline{A}$ is continuous (entrywise) on an interval $I = [t_0, T]$. Then ($\underline{1}$) has a unique solution for every choice of initial values $y(t_0) = y_0$ on the interval I.

If $\underline{F} \equiv 0$ (homogeneous ODE system) then the solution depends linearly on the initial values, in the sense that

$$y(t) = \underbrace{\underline{\underline{R}}(t, t_0)}_{\in\mathbb{R}^{n\times n}}\, y_0, \qquad t \in I$$

The matrix $\underline{\underline{R}}(t, t_0)$ is called the resolvent of the homogeneous ODE system $y'(t) = \underline{A}(t)\, y_0$.

Properties of the resolvent:

Consider $y'(t) = \underline{A}(t)\, y(t)$, $y(t_0) = y_0$ $\Rightarrow$ $y(t) = \underline{\underline{R}}(t, t_0)\, y_0$

① For $t_2 \geq t_1 \geq t_0$, $\underline{\underline{R}}(t_2,t_0) = \underline{\underline{R}}(t_2,t_1)\,\underline{\underline{R}}(t_1,t_0)$

and $\underline{\underline{R}}(t_0,t_0) = \underline{\underline{I}}$

Proof: Take $t_1 > t_0$, so that $y(t_1) = \underline{\underline{R}}(t_1,t_0)\, y_0 \equiv \tilde{y}$

Define $\underline{z}'(t) = \underline{\underline{A}}\,\underline{z}(t)$ with initial condition $\underline{z}(t_1) = \hat{y}$

Take $t_2 > t_1$: $\underline{z}(t_2) = \underline{\underline{R}}(t_2,t_1)\,\hat{y} = \underline{\underline{R}}(t_2,t_1)\, y(t_1)$

$$= \underline{\underline{R}}(t_2,t_1)\,\underline{\underline{R}}(t_1,t_0)\, y_0$$

But $y(t_2) = \underline{\underline{R}}(t_2,t_0)\, y_0$. By uniqueness of solutions (as the previous theorem stated), we also have $\underline{z}(t_2) = y(t_2)$

$$\Rightarrow \underline{\underline{R}}(t_2,t_0)\, y_0 = \underline{\underline{R}}(t_2,t_1)\,\underline{\underline{R}}(t_1,t_0)\, y_0 \quad \forall y_0 \in \mathbb{R}^n$$

$$\Leftrightarrow \underbrace{\left[\underline{\underline{R}}(t_2,t_0) - \underline{\underline{R}}(t_2,t_1)\,\underline{\underline{R}}(t_1,t_0)\right]}_{\text{Must be zeros for this to hold}} y_0 = 0 \quad \forall y_0 \in \mathbb{R}^n$$

$$\therefore \underline{\underline{R}}(t_2,t_0) = \underline{\underline{R}}(t_2,t_1)\,\underline{\underline{R}}(t_1,t_0) \qquad \text{(Semi-group property)}$$

Furthermore, $y(t_0) = \underline{\underline{R}}(t_0,t_0)\, y_0 = y_0 \Rightarrow \underline{\underline{R}}(t_0,t_0)$

② For $t_1 \geq t_0$, $(\underline{\underline{R}}(t_1,t_0))^{-1} = \underline{\underline{R}}(t_0,t_1)$

Proof: The previous proof works for $t_2 = t_0$ and $t_1 \geq t_0$:

$$\underbrace{\underline{\underline{R}}(t_0,t_0)}_{\underline{\underline{I}}} = \underline{\underline{R}}(t_0,t_1)\,\underline{\underline{R}}(t_1,t_0) \Rightarrow \underline{\underline{R}}(t_0,t_1)\,\underline{\underline{R}}(t_1,t_0) = \underline{\underline{I}}$$

$$\Rightarrow \quad \underline{R}(t_0, t_1) = \left(\underline{R}(t_1, t_0)\right)^{-1}$$

* * *

Formula for the resolvent

Consider a fundamental set of solutions $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n\}$ and define:

$$\underline{\underline{W}}(t) = \Big[\underbrace{\tilde{y}_1(t)\ \tilde{y}_2(t)\ \dots\ \tilde{y}_n(t)}_{} \Big]$$

Matrix whose columns are the fundamental solutions to $y'(t) = \underline{\underline{A}}(t)\, y(t)$.

Then $\underline{\underline{W}}'(t) = \underline{\underline{A}}(t)\, \underline{\underline{W}}(t)$. Moreover, since the general solution is a linear combination of $\{\tilde{y}_1, \dots, \tilde{y}_n\}$:

$$y(t) = C_1 \tilde{y}_1(t) + \dots + C_n \tilde{y}_n(t)$$

$$= \underline{\underline{W}}(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underline{\underline{W}}(t)\, \underline{c} \quad \text{with } \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

With the initial condition $y(t_0) = y_0 = \underline{\underline{W}}(t_0)\, \underline{c}$

$$\Rightarrow \underline{c} = \left(\underline{\underline{W}}(t_0)\right)^{-1} y_0$$

$$\Rightarrow y(t) = \underbrace{\underline{\underline{W}}(t)\left(\underline{\underline{W}}(t_0)\right)^{-1}}_{\underline{R}(t, t_0)} y_0$$

$$\therefore \quad \boxed{\underline{R}(t, t_0) = \underline{\underline{W}}(t)\left(\underline{\underline{W}}(t_0)\right)^{-1}}$$

Inhomogeneous systems ($\underline{F}(t) \neq 0$).

Suppose $y'(t) = \underline{\underline{A}}(t)\,\underline{y}(t) + \underline{F}(t)$ and that the solution of the homogeneous system is $\underline{y}_H(t) = \underline{\underline{W}}(t)\,\underline{c}$.

Variation of constant : $y(t) = \underline{\underline{W}}(t)\,\underline{c}(t)$ ($\underline{c}$ varies with t now.)

Differentiate : $$y'(t) = \underline{\underline{W}}'(t)\,\underline{c}(t) + \underline{\underline{W}}(t)\,\underline{c}'(t)$$

$$= \underline{\underline{A}}(t)\underline{\underline{W}}(t)\,\underline{c}(t) + \underline{\underline{W}}(t)\,c'(t) \qquad (\underline{\underline{W}}'(t) = \underline{\underline{A}}(t)\underline{\underline{W}}(t))$$

$$= \underline{\underline{A}}(t)\,\underline{y}(t) + \underline{\underline{W}}(t)\,\underline{c}'(t)$$

$$= \underline{\underline{A}}(t)\,y(t) + \underline{F}(t)$$

Set $\underline{\underline{W}}(t)\,\underline{c}'(t) = \underline{F}(t)$

$$\Rightarrow \boxed{\underline{c}'(t) = (\underline{\underline{W}}(t))^{-1}\,\underline{F}(t)}$$

Integrate : $$\underline{c}(t) = \underline{c}(t_0) + \int_{t_0}^{t} (\underline{\underline{W}}(s))^{-1}\,\underline{F}(s)\,ds$$

Initial condition : $y(t_0) = y_0 = \underline{\underline{W}}(t_0)\,\underline{c}(t_0) \Rightarrow \underline{c}(t_0) = (\underline{\underline{W}}(t_0))^{-1}\,y_0$.

$$\hookrightarrow \underline{c}(t) = (\underline{\underline{W}}(t_0))^{-1}\,y_0 + \int_{t_0}^{t} (\underline{\underline{W}}(s))^{-1}\,\underline{F}(s)\,ds$$

$$\Rightarrow y(t) = \underbrace{\underline{\underline{W}}(t)(\underline{\underline{W}}(t_0))^{-1}}_{\underline{\underline{R}}(t,t_0)}\,y_0 + \int_{t_0}^{t} \underbrace{\underline{\underline{W}}(t)(\underline{\underline{W}}(s))^{-1}}_{\underline{\underline{R}}(t,s)}\,\underline{F}(s)\,ds$$

Writing $y(t)$ in terms of the resolvent, we finally get

$$\boxed{y(t) = \underline{\underline{R}}(t,t_0)\,y_0 + \int_{t_0}^{t} \underline{\underline{R}}(t,\tau)\,\underline{F}(\tau)\,d\tau}$$

↳ (This procedure and formula is known as Duhamel's principle.)

There also holds:

$$\det(\underline{\underline{W}}(t)) = \det(\underline{\underline{W}}(t_0))\,e^{\int_{t_0}^{t} \operatorname{tr}(\underline{\underline{A}}(\tau))\,d\tau},$$

where $\operatorname{tr}(\underline{\underline{A}})$ = sum of the diagonal elements of $\underline{\underline{A}}(t)$

$$= a_{11}(t) + \cdots + a_{nn}(t) = \sum_{k=1}^{n} (\underline{\underline{A}}(t))_{kk}.$$

This is known as the Abel-Liouville-Jacobi-Ostrogradskii identity (or Liouville's identity)

§ 3.4. Solution methods when the matrix $\underline{\underline{A}}$ is constant

We will consider the first-order ODE system (or dynamical system)

$$y'(t) = \underline{\underline{A}}\,y(t) + \underline{F}(t), \quad y(t_0) = y_0$$

↳ $\underline{\underline{A}}$ is constant.

§3.4.1 Solution by diagonalization

- Suppose that $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ is diagonalizable:

$$\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} := \underline{\underline{D}} \;, \text{ where } \underline{\underline{T}} = \underbrace{\left[\underline{v}_1 \ \underline{v}_2 \ \cdots \ \underline{v}_n \right]}_{\text{matrix whose columns are the eigenvectors of } \underline{\underline{A}}.}$$

Then $\underline{\underline{A}} = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}$ by inverting the matrices.

- Consider the homogeneous ODE system $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$.

As $\underline{\underline{A}} = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}$, we can write the ODE system as $\underline{y}'(t) = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}\underline{y}(t)$,

or $\left(\underline{\underline{T}}^{-1}\underline{y}'(t)\right) = \underline{\underline{D}}\left(\underline{\underline{T}}^{-1}\underline{y}(t)\right)$.

- Substitute $\underline{z}(t) = \underline{\underline{T}}^{-1}\underline{y}(t)$ in the ODE to get

$$\underline{z}'(t) = \underline{\underline{D}}\,\underline{z}(t).$$

In component form,

$$\begin{pmatrix} z_1'(t) \\ z_2'(t) \\ \vdots \\ z_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n'(t) \end{pmatrix} \leftrightarrow \begin{array}{c} z_j'(t) = \lambda_j z_j(t) \\ j = 1, \dots, n. \end{array}$$

This gives a decoupled system of n ODEs $z_j'(t) = \lambda_j z_j(t)$, $j = 1, \dots, n$.

$\Rightarrow$ Solve the n ODEs separately!

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Writing $\underline{y}(t)$ in terms of the resolvent, we finally get

$$\boxed{\underline{y}(t) = \underline{\underline{R}}(t,t_0)\underline{y}_0 + \int_{t_0}^{t} \underline{\underline{R}}(t,\tau)\underline{F}(\tau)\,d\tau}$$

↳ (This procedure and formula is known as Duhamel's principle.)

There also holds:

$$\det(\underline{\underline{W}}(t)) = \det(\underline{\underline{W}}(t_0))\, e^{\int_{t_0}^{t} \operatorname{tr}(\underline{\underline{A}}(\tau))\,d\tau},$$

where $\operatorname{tr}(\underline{\underline{A}})$ = sum of the diagonal elements of $\underline{\underline{A}}(t)$

$$= a_{11}(t) + \cdots + a_{nn}(t) = \sum_{k=1}^{n} (\underline{\underline{A}}(t))_{kk}.$$

This is known as the Abel-Liouville-Jacobi-Ostrogradskii identity (or Liouville's identity)

§3.4. Solution methods when the matrix $\underline{\underline{A}}$ is constant

We will consider the first-order ODE system (or dynamical system)

$$\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t) + \underline{F}(t), \quad \underline{y}(t_0) = \underline{y}_0$$

↳ $\underline{\underline{A}}$ is constant.

§3.4.1 Solution via diagonalization

- Suppose that $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ is diagonalizable:

$$\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix} := \underline{\underline{D}}, \text{ where } \underline{\underline{T}} = \underbrace{\left[\underline{v}_1\ \underline{v}_2\ \cdots\ \underline{v}_n\right]}_{\text{Matrix whose columns are the eigenvectors of } \underline{\underline{A}}.}$$

Then $\underline{\underline{A}} = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}$ by inverting the matrices.

- Consider the homogeneous ODE system $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$.

As $\underline{\underline{A}} = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}$, we can write the ODE system as $\underline{y}'(t) = \underline{\underline{T}}\,\underline{\underline{D}}\,\underline{\underline{T}}^{-1}\underline{y}(t)$,

or $\left(\underline{\underline{T}}^{-1}\underline{y}'(t)\right) = \underline{\underline{D}}\left(\underline{\underline{T}}^{-1}\underline{y}(t)\right)$.

- Substitute $\underline{z}(t) = \underline{\underline{T}}^{-1}\underline{y}(t)$ in the ODE to get

$\underline{z}'(t) = \underline{\underline{D}}\,\underline{z}(t)$. In component form,

$$\begin{pmatrix} z_1'(t)\\ z_2'(t)\\ \vdots \\ z_n'(t)\end{pmatrix} = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}\begin{pmatrix} z_1(t)\\ z_2(t)\\ \vdots \\ z_n'(t)\end{pmatrix} \leftrightarrow z_j'(t) = \lambda_j z_j(t),\ j=1,\dots,n.$$

This gives a decoupled system of n ODEs $z_j'(t) = \lambda_j z_j(t)$, $j=1,\dots,n$.

$\Rightarrow$ Solve the n ODEs separately!

Since $z_j'(t) = \lambda_j z_j(t)$, $j=1,\dots,n$, the solutions are given by

$$z_j(t) = C_j e^{\lambda_j t}, \quad j=1,\dots,n.$$

$$\Rightarrow \underbrace{\begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}}_{=\underline{z}(t)} = \underbrace{\begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}}_{:=\underline{\underline{E}}(t)} \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}}_{:=\underline{C}}$$

- Now we use that $\underline{y}(t) = \underline{\underline{T}}\,\underline{z}(t)$:

$$\Rightarrow \underline{y}(t) = \underline{\underline{T}}\,\underline{\underline{E}}(t)\,\underset{\uparrow}{\underline{c}}$$

- Initial condition $\underline{y}(t_0) = \underline{y}_0 = \underline{\underline{T}}\,\underline{\underline{E}}(t_0)\,\underline{c}$

$$\Rightarrow \underline{c} = \left(\underline{\underline{E}}(t_0)\right)^{-1}\left(\underline{\underline{T}}\right)^{-1}\underline{y}_0 \text{, and note that}$$

$$\left(\underline{\underline{E}}(t_0)\right)^{-1} = \begin{pmatrix} e^{\lambda_1 t_0} & & & \\ & e^{\lambda_2 t_0} & & \\ & & \ddots & \\ & & & e^{\lambda_n t_0} \end{pmatrix}^{-1} = \begin{pmatrix} e^{-\lambda_1 t_0} & & & \\ & e^{-\lambda_2 t_0} & & \\ & & \ddots & \\ & & & e^{-\lambda_n t_0} \end{pmatrix}$$

$$\therefore \underline{y}(t) = \underline{\underline{T}}\left(\underline{\underline{E}}(t)\,\underline{\underline{E}}(t_0)^{-1}\right)\underline{\underline{T}}^{-1}\underline{y}_0$$

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Summary: ODE: $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t) + \underline{F}(t)$

① Find the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and their corresponding eigenvectors eigenvectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ of the matrix $\underline{\underline{A}}$.

② Then let $\underline{\underline{T}} = [\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_n]$, $\underline{\underline{E}}(t) = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix}$

and compute the inverse of $\underline{\underline{T}}$, which we will denote by $\underline{\underline{T}}^{-1}$.

F ≡ 0 ③ Then $\underline{y}(t) = \underline{\underline{T}}\,\underline{z}(t)$ where $\underline{z}(t) = \underline{\underline{E}}(t)\,\underline{c}$, where

$\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is a vector of coefficients.

$\Rightarrow \underline{y}(t) = \underline{\underline{T}}\,\underline{\underline{E}}(t)\,\underline{c}$, and with $\underline{y}(t_0) = \underline{y}_0$, we find $\underline{y}(t) = \underline{\underline{T}}\left(\underline{\underline{E}}(t)\,\underline{\underline{E}}(t_0)^{-1}\right)\underline{\underline{T}}^{-1}\,\underline{y}_0$.

F ≢ 0 ④ If $F \not\equiv 0$, we write $\underline{y}(t) = \underline{\underline{T}}\,\underline{\underline{E}}(t)\,\underline{c}(t)$ and use the method of variation of constants to find $\underline{c}(t)$.

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, $y(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

① Eigenvalues & Eigenvectors of $\underline{\underline{A}}$: $\lambda_1 = 2$, $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\lambda_2 = 3$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② $\underline{\underline{T}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\underline{\underline{E}}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$, and so $\underline{z}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Hence $\underline{y}(t) = \begin{pmatrix}2&1\\1&1\end{pmatrix}\underline{z}(t) = \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$, and $\underline{y}(0)=\begin{pmatrix}4\\3\end{pmatrix}$.

Since $\underline{y}(0) = \underline{\underline{T}}\begin{pmatrix}c_1\\c_2\end{pmatrix} \Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \underline{\underline{T}}^{-1}\underline{y}(0) = \begin{pmatrix}2&1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}4\\3\end{pmatrix}$,

$$\underline{y}(t) = \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}4\\3\end{pmatrix}$$

Here, $\begin{pmatrix}2&1\\1&1\end{pmatrix}^{-1} = \begin{pmatrix}1&-1\\-1&2\end{pmatrix}$, and

$$\begin{aligned}\underline{y}(t) &= \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}1&-1\\-1&2\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\\ &= \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}e^{2t}\\2e^{3t}\end{pmatrix} = \begin{pmatrix}2\left[e^{2t}+e^{3t}\right]\\e^{2t}+2e^{3t}\end{pmatrix}\\ &= \begin{pmatrix}2\\1\end{pmatrix}e^{2t} + 2\begin{pmatrix}1\\1\end{pmatrix}e^{3t}.\end{aligned}$$

§ 3.4.2. Solution via Matrix exponentials

Consider the ODE $y'(t) = a\,y(t)$. Its solution is $y(t) = Ce^{at}$.

Question: $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t) \overset{?}{\Rightarrow} \underline{y}(t) = \underbrace{\left(e^{\underline{\underline{A}}t}\right)}_{\text{Exponential of a matrix ?}}\underline{c}$

Def^n: For $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, the matrix exponential is given by

$$e^{\underline{\underline{A}}} = \sum_{k=0}^{\infty}\frac{(\underline{\underline{A}})^k}{k!} \in \mathbb{R}^{n\times n}, \quad \underline{\underline{A}}^k = \underbrace{\underline{\underline{A}}\cdot\underline{\underline{A}}\cdot\ldots\cdot\underline{\underline{A}}}_{k\text{-times}}$$

and $\underline{\underline{A}}^0 = \underline{\underline{I}} \leftarrow$ Identity matrix.

Remark 1:

(i) In general, if $(\underline{\underline{A}})_{ij} = a_{ij}$, $i,j = 1,\dots,n$, then we DO NOT generally have that $(e^{\underline{\underline{A}}})_{ij} = e^{a_{ij}}$, that is,

$$e^{\underline{\underline{A}}} \neq \begin{pmatrix} e^{a_{11}} & e^{a_{12}} & \cdots & e^{a_{1n}} \\ e^{a_{21}} & e^{a_{22}} & & \\ \vdots & & \ddots & \\ e^{a_{n1}} & & & e^{a_{nn}} \end{pmatrix} \text{ in general.}$$

(ii) Given $\underline{\underline{A}}, \underline{\underline{B}} \in \mathbb{R}^{n\times n}$, then $e^{\underline{\underline{A}}} \cdot e^{\underline{\underline{B}}} = e^{\underline{\underline{A}}+\underline{\underline{B}}}$ if and only if $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$, i.e., the matrices commute.

(iii) If $\underline{\underline{D}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ is an $n\times n$ diagonal matrix, then in fact

$$e^{\underline{\underline{D}}} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & \ddots & e^{\lambda_n} \end{pmatrix}$$

Indeed, $e^{\underline{\underline{D}}} = \sum_{k=0}^{\infty} \frac{(\underline{\underline{D}})^k}{k!}$ and $\underline{\underline{D}}^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & \ddots & \lambda_n^k \end{pmatrix}$, so that

$$e^{\underline{\underline{D}}} = \sum_{k=0}^{\infty} \begin{pmatrix} \left(\frac{\lambda_1^k}{k!}\right) & & 0 \\ & \left(\frac{\lambda_2^k}{k!}\right) & \\ 0 & \ddots & \left(\frac{\lambda_n^k}{k!}\right) \end{pmatrix} = \begin{pmatrix} \left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!}\right) & & 0 \\ & \left(\sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!}\right) & \\ 0 & & \left(\sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!}\right) \end{pmatrix}$$

and since $\sum_{k=0}^{\infty} \frac{\lambda_j^k}{k!} = e^{\lambda_j}$, $j=1,\dots,n$, we get $e^{\underline{D}} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots\ e^{\lambda_n} \end{pmatrix}$.

Ex: $\underline{A} = \begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix}$, which is not diagonalizable.

$$\underline{A}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underline{A}^1 = \begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix}, \quad \underline{A}^2 = \begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix}\begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow A^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } k \geq 2$$

So $$e^{\underline{A}t} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \underline{I} + At + \underbrace{\sum_{k=2}^{\infty} \frac{(At)^k}{k!}}_{=0}$$

$$\hookrightarrow = \begin{pmatrix} 1-5t & t \\ -25t & 1+5t \end{pmatrix}$$

Remark 2: If the matrix $\underline{A}$ is diagonalizable, then

$$\underline{T}^{-1}\underline{A}\,\underline{T} = \underline{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{pmatrix} \text{ and } e^{(\underline{T}^{-1}\underline{A}\,\underline{T})} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots\ e^{\lambda_n} \end{pmatrix} = e^{\underline{D}}$$

However, $e^{\underline{T}^{-1}\underline{A}\,\underline{T}} = \sum_{k=0}^{\infty} \frac{(\underline{T}^{-1}\underline{A}\,\underline{T})^k}{k!}$ and $(\underline{T}^{-1}\underline{A}\,\underline{T})^k = \underline{T}^{-1}\underline{A}^k\underline{T}$. Indeed,

$$(\underline{T}^{-1}\underline{A}\,\underline{T})^k = (\underline{T}^{-1}\underline{A}\,\underbrace{\underline{T})\cdot(\underline{T}^{-1}}_{\underline{T}\,\underline{T}^{-1} = \underline{I}\ (!)}\underline{A}\,\underline{T})\cdot \ldots \cdot(\underline{T}^{-1}\underline{A}\,\underline{T}) \quad (k \text{ times})$$

$$\hookrightarrow = \underline{T}^{-1}\underline{A}^k\underline{T}.$$

Whence $e^{\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}}} = \sum_{k=0}^{\infty} \frac{(\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}})^k}{k!} = \sum_{k=0}^{\infty} \left(\underline{\underline{T}}^{-1} \frac{\underline{\underline{A}}^k}{k!} \underline{\underline{T}}\right) = \underline{\underline{T}}^{-1}\left(\sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k}{k!}\right)\underline{\underline{T}}$

$= \underline{\underline{T}}^{-1} e^{\underline{\underline{A}}}\, \underline{\underline{T}}$. Since $e^{\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}}} = e^{\underline{\underline{D}}} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$, we get

$$e^{\underline{\underline{A}}} = \underline{\underline{T}}\, e^{\underline{\underline{D}}}\, \underline{\underline{T}}^{-1} = \underline{\underline{T}} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} \underline{\underline{T}}^{-1}.$$

* * *

- How do we use matrix exponentials to solve systems of the form $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$?

Theorem: Define $\underline{\underline{\Phi}} : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ as $\underline{\underline{\Phi}}(t) := e^{\underline{\underline{A}}t}$.

Then $\underline{\underline{\Phi}}'(t) = \underline{\underline{A}}\,\underline{\underline{\Phi}}(t)$.

Proof: $\underline{\underline{\Phi}}'(t) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} (\underline{\underline{A}}t)^k/k!\right) = \sum_{k=1}^{\infty} \frac{k t^{k-1} \underline{\underline{A}}^k}{k!}$,

$$= \underline{\underline{A}}\left(\sum_{k=1}^{\infty} \frac{t^{k-1}\underline{\underline{A}}^{k-1}}{(k-1)!}\right) = \underline{\underline{A}}\left(\sum_{k=0}^{\infty} (\underline{\underline{A}}t)^k/k!\right),$$

$$= \underline{\underline{A}}\, e^{\underline{\underline{A}}t} = \underline{\underline{A}}\,\underline{\underline{\Phi}}(t).$$

∎

If we write $\underline{\underline{\Phi}}(t) = \begin{bmatrix} \top & \top & & \top \\ \underline{y}_1(t) & \underline{y}_2(t) & \cdots & \underline{y}_n(t) \\ \bot & \bot & & \bot \end{bmatrix}$, then the theorem says that $\underline{y}_j'(t) = \underline{\underline{A}}\,\underline{y}_j(t)$.

=> The columns of the matrix exponential $e^{\underline{\underline{A}}t}$ are fundamental solutions to $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$. As any linear combination of $\underline{y}_1, \dots, \underline{y}_n$ is also a solution, the general solution is $\underline{y}(t) = e^{\underline{\underline{A}}t}\cdot \underline{c}$, $\underline{c} \in \mathbb{R}^n$.

$\Rightarrow$ If $\underline{y}(t_0) = \underline{y}_0$, then $\underline{y}_0 = e^{\underline{\underline{A}} t_0} \cdot \underline{c} \Rightarrow \underline{c} = (e^{\underline{\underline{A}} t_0})^{-1} \underline{y}_0$.

But $(e^{\underline{\underline{A}} t_0})^{-1} = e^{-\underline{\underline{A}} t_0}$ as $\underline{\underline{A}} \cdot (-\underline{\underline{A}}) = (-\underline{\underline{A}})(\underline{\underline{A}})$ so that $e^{\underline{\underline{A}} - \underline{\underline{A}}} = \underline{\underline{I}} = e^{\underline{\underline{A}}} \cdot e^{-\underline{\underline{A}}}$
$\Rightarrow (e^{\underline{\underline{A}}})^{-1} = e^{-\underline{\underline{A}}}$.

$$\Rightarrow \underline{y}(t) = e^{\underline{\underline{A}}(t-t_0)} \cdot \underline{y}_0 .$$

Ex: $\underline{\underline{A}} = \begin{pmatrix} -5 & 1 \\ -25 & 5 \end{pmatrix}$, which is not diagonalizable. In the previous example, we computed its matrix exponential $e^{\underline{\underline{A}}t} = \begin{pmatrix} 1-5t & t \\ -25t & 1+5t \end{pmatrix}$.

If $\underline{y}(0) = \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \underline{y}_0 \Rightarrow \underline{y}(t) = e^{\underline{\underline{A}}t} \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1-5t & t \\ -25t & 1+5t \end{pmatrix} \begin{pmatrix} -3 \\ 5 \end{pmatrix}$
$= \begin{pmatrix} 20t - 3 \\ 100t + 5 \end{pmatrix}$.

* * *

§3.4.3. Solution by Laplace transforms

Consider the system $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t) + \underline{F}(t)$, $\underline{y}(0) = \underline{y}_0$.

Strategy: $\underline{y}(t) = (y_1(t), \dots, y_n(t))$

$$\begin{matrix} y_1(t) & \dots & y_n(t) \\ \downarrow & & \downarrow \\ Y_1(s), & \dots, & Y_n(s) \end{matrix}$$

- Apply the Laplace transform to each of the equations in (1) separately.
- Obtain a system of linear equations $(n \times n)$ for $Y_1(s), \dots, Y_n(s)$.
- Solve the system for $Y_1(s), \dots, Y_n(s)$, each in term of s.

Take the inverse Laplace transform of $Y_1(s),\dots,Y_n(s) \to y_1(t),\dots,y_n(t)$.

Ex: $\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_n(t) \end{pmatrix}$, $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

Then $\mathcal{L}[y_1'](s) = sY_1(s) - y_1(0) = sY_1(s) - \alpha$ and
$\mathcal{L}[y_2'](s) = sY_2(s) - y_2(0) = sY_2(s) - \beta$.

$$\Rightarrow \begin{cases} sY_1(s) - \alpha = aY_1(s) + bY_2(s) \\ sY_2(s) - \beta = cY_1(s) + dY_2(s) \end{cases}$$

$$\Rightarrow \begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Solve this system to get $Y_1(s)$ and $Y_2(s)$, and invert the result to get $y_1(t)$ & $y_2(t)$!

§3.5 Stability

Consider the system $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$, $\underline{\underline{A}}$ is constant.

Defn: A solution of this system is called an equilibrium solution (and denoted by $\underline{y}^*$) if it is constant ($\frac{d\underline{y}^*}{dt} = 0$). Such solutions are also called critical points or fixed points.

Questions:
- What is the behavior of solutions near an equilibrium point?
- Do solutions tend to a stationary state as $t \to \infty$? That is, what is $\lim_{t\to\infty} \underline{y}(t)$?

Ex: $\underline{y}'(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \underline{y}(t) \Rightarrow \underline{y}^* = \underline{0}$ is the only equilibrium solution and as

$\underline{y}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ $\forall C_1, C_2 \in \mathbb{R}$, we find $\lim_{t\to\infty} \underline{y}(t) = \underline{0}$.

Remarks: 1) If $\det(\underline{\underline{A}}) \neq 0$ then the only equilibrium solution to $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$ is $\underline{y}^* = 0$.

From now on we assume that $\underline{\underline{A}} \in \mathbb{R}^{2\times2}$ and $\det \underline{\underline{A}} \neq 0$.

2) We will denote by λ_1 & λ_2 the eigenvalues of $\underline{\underline{A}}$ and by v_1 & v_2 their respective eigenvectors.

If $\underline{\underline{A}}$ is diagonalizable, then remember that the solution $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$, $\underline{y}(t_0) = \underline{y}_0$ is given by

$$\underline{y}(t) = [\underline{\underline{T}}\,\underline{\underline{E}}(t)\,\underline{\underline{E}}(t_0)\,\underline{\underline{T}}^{-1})\,\underline{y}_0, \text{ where } \underline{\underline{T}} = (\underline{v}_1, \underline{v}_2) \;\&\; \underline{\underline{E}}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

The long-term behavior of $\underline{y}(t)$ (with $\underline{y}_0 \neq \underline{0}$) is therefore prescribed by the eigenvalues and eigenvectors of $\underline{\underline{A}}$.

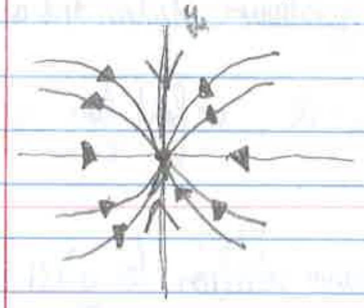
For instance, if both $\lambda_1 < \lambda_2 < 0$ then $\underline{\underline{E}}(t) \to \underline{\underline{0}}$ as $t \to \infty$ and $\underline{y}(t) \to \underline{0}$ identically. We will classify the dynamics of trajectories near the equilibrium solution $\underline{y}^* = \underline{0}$ in terms of the eigenvalues.

Before we do 20, let's give an instructive example

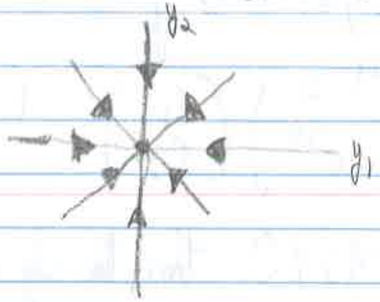
Ex: $\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $y(0) = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$

Solution : $y_1(t) = y_{01} e^{at}$
$y_2(t) = y_{02} e^{-t}$.

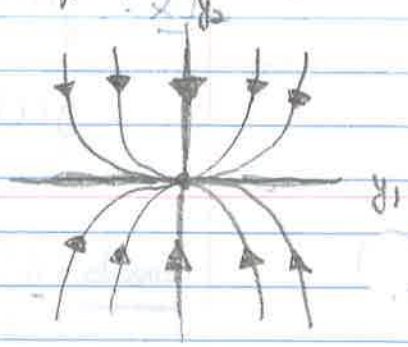
Let's draw the phase portrait (plot of y_2 vs y_1) for different values of a :



$a < -1$



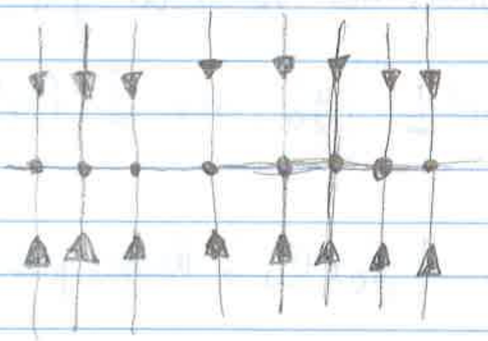
$a = -1$



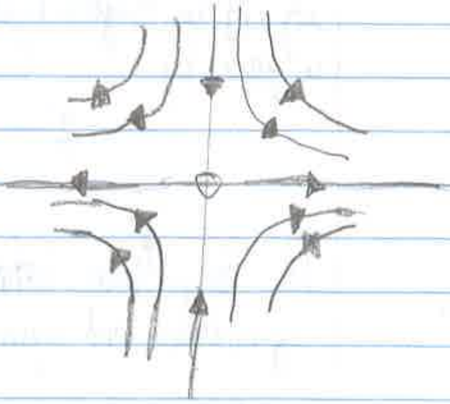
$-1 < a < 0$

When $a < 0$, $y_1(t) \to 0$ as $t \to \infty$ and ~~∞~~ $y(t) \to \underline{0}$ as $t \to \infty$. However, the direction of approach depends on the size of a compared to -1.

If $a = 0$ or $a > 0$:



$a = 0$



$a > 0$

If $a = 0$, then $y_1(t) = y_{01}$ and there is a line of equilibrium points on $y_2(t) = 0$. All trajectories approach these fixed points along vertical lines.

If $a > 0$, $\underline{y}^* \equiv 0$ becomes unstable and trajectories veer away from $\underline{y}^* \equiv 0$ and head out to $\lim_{t\to\infty} y_1(t) = 0$ and $\lim_{t\to\infty} y_2(t) = \pm\infty$. The only exception is when $y_{02} = 0$ and we are on the $y_1(t) = \underline{0}$ axis; then it walks a tight rope to the origin.

- We have different types of equilibrium solution in this example:

If $a < -1$ or $-1 < a < 0$ we call $\underline{y}^* \equiv 0$ a <u>stable node</u>
If $a = -1$ " " a <u>stable star</u>
If $a > 0$ " " a <u>saddle point</u>
If $a = 0$ we call $\begin{pmatrix} y_1(t) \\ 0 \end{pmatrix}$ a <u>line of fixed points</u>.

~~~

Suppose now that $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Its eigenvalues are given by the characteristic polynomial $\det|\underline{\underline{A}} - \underline{\underline{I}}\lambda| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow \lambda^2 - (a+d)\lambda + (ab - cd) = 0$$

$$\Leftrightarrow \lambda^2 - \operatorname{tr}\underline{\underline{A}}\,\lambda + \det\underline{\underline{A}} = 0$$

, where $\operatorname{tr}\underline{\underline{A}} = (a+d)$ (sum of diagonal elements)

$\det\underline{\underline{A}} = ad - bc$ (determinant)

Eigenvalues: $$\lambda_1 = \frac{\operatorname{tr}\underline{\underline{A}} + \sqrt{(\operatorname{tr}\underline{\underline{A}})^2 - 4\det\underline{\underline{A}}}}{2}$$

$$\lambda_2 = \frac{\operatorname{tr}\underline{\underline{A}} - \sqrt{(\operatorname{tr}\underline{\underline{A}})^2 - 4\det\underline{\underline{A}}}}{2}$$

If $\lambda_1 \neq \lambda_2$ then $\exists$ linearly independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ that form a basis of $\mathbb{R}^2$. If we have the system $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$, $\underline{y}(t_0) = \underline{y_0}$
~~~

then we may write $\underline{y}_0 = C_1 \underline{v_1} + C_2 \underline{v_2}$ and the corresponding solution is

$$\underline{y}(t) = C_1 e^{\lambda_1 t} \underline{v_1} + C_2 e^{\lambda_2 t} \underline{v_2} .$$

(If $\lambda_1 = \lambda_2$, we may still find two linearly independent eigenvectors, in which case

$$\underline{y}(t) = C_1 e^{\lambda_1 t} \underline{v_1} + C_2 e^{\lambda_1 t} \underline{v_2} .$$

If not, then the solution is $\underline{y}(t) = C_1 e^{\lambda_1 t} \underline{v_1} + C_2 e^{\lambda_1 t}(t v_1 + \underline{w})$ for some vector $\underline{w}$.)

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $\operatorname{tr} \underline{\underline{A}} = -1$, $\det \underline{\underline{A}} = -6$. Then

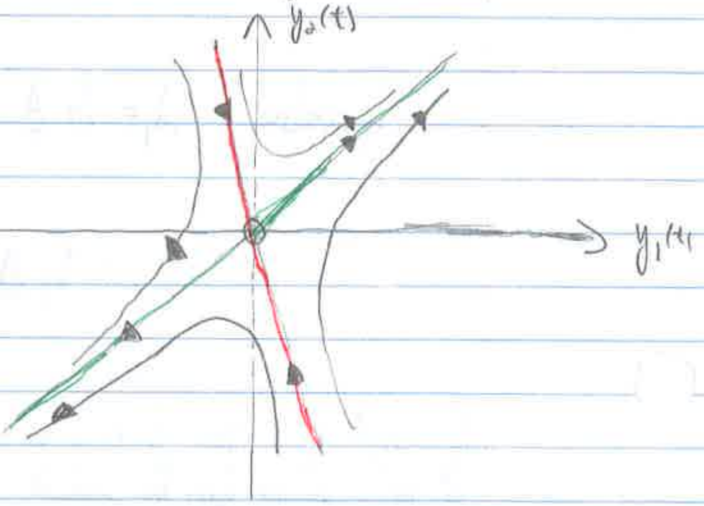
$$(\operatorname{tr} \underline{\underline{A}})^2 - 4 \det \underline{\underline{A}} = 1 + 24 = 25, \text{ and}$$

$$\lambda_1 = \frac{-1 + \sqrt{25}}{2} = 2, \quad \lambda_2 = \frac{-1 - \sqrt{25}}{2} = -3$$

The corresponding eigenvectors are $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\underline{v_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Here, the first eigensolution of $\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t)$, $\underline{y}(t_0) = \underline{y}_0 = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$ (given by $C_1 e^{\lambda_1 t} \underline{v_1}$) grows exponentially, and the second eigensolution $C_2 e^{\lambda_2 t} \underline{v_2}$ decays. This means the origin is a saddle point.

The corresponding phase portrait is

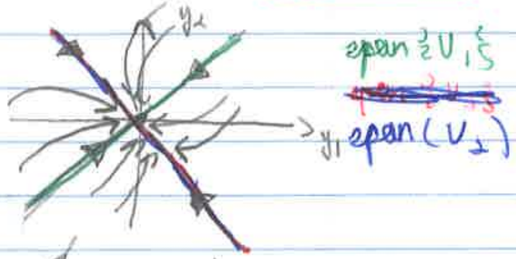


We call here span$\{\underline{v_1}\}$ the unstable manifold of the equilibrium solution $\underline{y}^* \equiv \underline{0}$ and we call span$\{\underline{v_2}\}$ the stable manifold of the equilibrium solution $\underline{y}^* \equiv \underline{0}$.

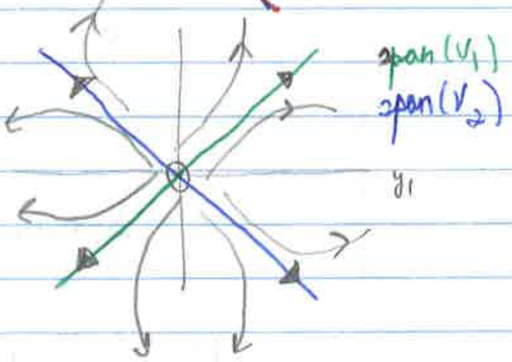
Let's finally tackle the general case.

Defn: Let $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have eigenvalues λ_1, λ_2 and respective eigenvectors v_1 & v_2. Consider the system $y'(t) = \underline{\underline{A}}\, y(t)$ where $\det \underline{A} \neq 0$ and let $y^* \equiv \underline{0}$ be the (only) equilibrium solution of this system.

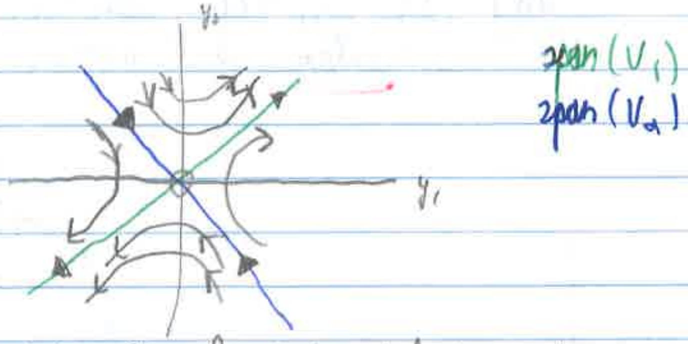
i) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2 < 0$ then $y^* \equiv \underline{0}$ is called a stable node.



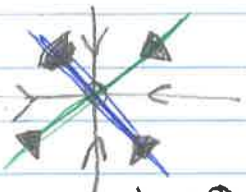
ii) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 > \lambda_2 > 0$ then $y^* \equiv \underline{0}$ is called an unstable node.



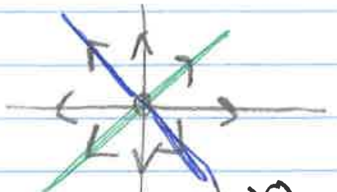
iii) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_2 < 0 < \lambda_1$ then $y^* \equiv \underline{0}$ is called a saddle point.



iv) If $\lambda_1 = \lambda_2 \in \mathbb{R}$ and there exists linearly independent eigenvectors v_1 and v_2, then $y^* \equiv \underline{0}$ is called a stable $(\lambda_1 < 0)$ or unstable $(\lambda_1 > 0)$ star node.



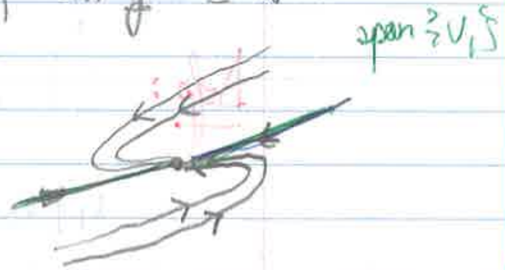
$\lambda_1 < 0$



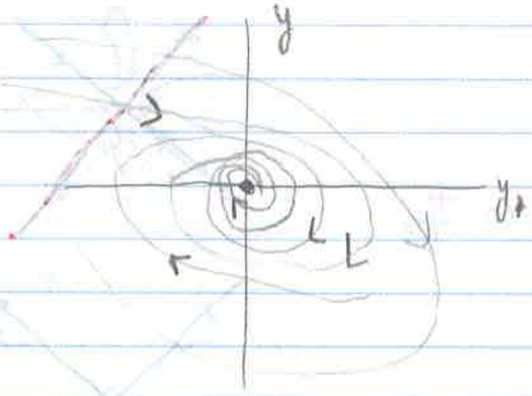
$\lambda_1 > 0$

v) If $\lambda_1 = \lambda_2 \in \mathbb{R}$ and only one eigenvector exists, then $y^* \equiv \underline{0}$ is called a degenerate node (

If $\lambda_1 < 0$, then the dynamics of trajectories around the node will look like this:



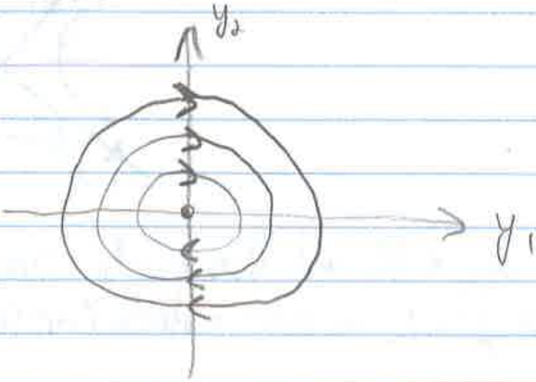
vi) If $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 + \lambda_2 \neq 0$, then $y^* \equiv \underline{0}$ is called a stable ($\lambda_1 + \lambda_2 < 0$) or unstable ($\lambda_1 + \lambda_2 > 0$) spiral.



$\lambda_1 + \lambda_2 < 0$

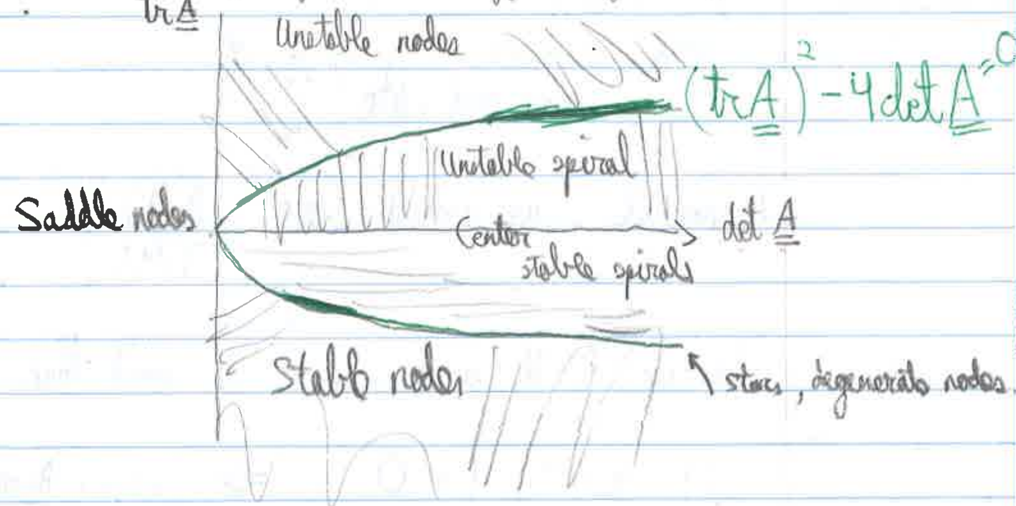
$\lambda_1 + \lambda_2 > 0$

vii) If $\lambda_1 = ib$ with $b \neq 0$ ($\lambda_2 = -ib$), then $y^* \equiv \underline{0}$ is called a center. The trajectories on the phase planes are circles



Since the eigenvalues λ_1 and λ_2 are given by $\dfrac{\operatorname{tr}\underline{\underline{A}} \pm \sqrt{(\operatorname{tr}\underline{\underline{A}})^2 - 4\det\underline{\underline{A}}}}{2}$,

we can show the type and stability of all the different fixed points on a single diagram:



Ex (Love affairs - Strogatz 1982)

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo tends to echo her; he warms up when she loves him, and grows cold when she hates him.

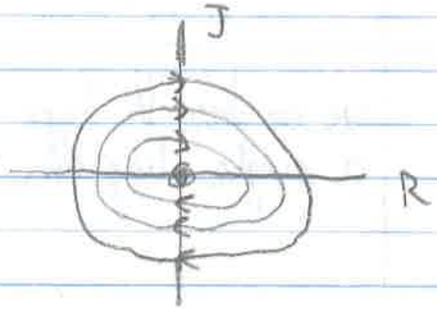
Let $R(t)$ = Romeo's love / hate for Juliet at time t.
$J(t)$ = Juliet's love / hate for Romeo at time t.

Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\begin{aligned} R'(t) &= a\,J(t) \qquad && a > 0 \\ J'(t) &= -b\,R(t) \qquad && b > 0 \end{aligned}$$

So $\underline{\underline{A}} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$ and $\lambda_1 = i\sqrt{ab}$, $\lambda_2 = -i\sqrt{ab}$.

↳ $y^* \equiv \underline{0}$ is a center.



Outcome: Neverending cycle of love and hate. ☹

In general, we would have $\begin{pmatrix} R'(t) \\ J'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} R(t) \\ J(t) \end{pmatrix}$.

The sign of the numbers a, b, c, d would then dictate their "romantic types";

1) $a > 0$, $b > 0$ => "Eager Beaver"
2) $a > 0$, $b < 0$ => "Narcissitic Nerd"
3) $a < 0$, $b > 0$ => "Bashful Budder"
4) $a < 0$, $b < 0$ => "Malevolent Misanthrope"

Can a "narcissitic nerd" find love with a "bashful budder"? Figure it out by classifying the resultant equilibrium point!

24 July 2019

§4 Nonlinear ODEs

What can we say about nonlinear ODEs and their solutions in general?
Suppose we have the ODE

$$y'(t) = 5t^2 \sin(y(t)) + \sqrt{y(t)^5 - 3t}$$
$$y(0) = 2.$$

What can be said about the solution of this ODE (if it exists)?

- Do solutions exist?
- If so, when is the solution unique?
- If solutions exist, do they exist for all $t \geq 0$ or for some interval $[0,T]$?

As we will see, these are not trivial questions!

Ex 1: The solution to $y'(t) = y^{1/3}(t)$ starting at $y(0) = 0$ is not unique.

A first obvious solution is $y(t) = 0$. We can get another if we separate variables and integrate:

$$\int \frac{1}{y^{1/3}}\, dy = \int 1\, dt \iff \frac{y^{2/3}(t)}{2/3} = t + C.$$

Since $y(0) = 0$ we get $C = 0$, whence $y(t) = \left(\frac{2t}{3}\right)^{3/2}$ is another solution.

$\therefore$ The ODE has more than one solution!

Ex 2: The solution to $y'(t) = 1 + y(t)^2$ starting at $y(0) = 0$ exists and is unique, but exists only in $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Separate variables and integrate: $\displaystyle\int \frac{1}{1+y^2}\, dy = t + C$

$\Leftrightarrow$ $\arctan(y(t)) = t + C$, and as $\arctan(C) = 0$,

$\arctan(y(0)) = C$ with $y(0) = 0 \Rightarrow C = 0$.

$\therefore$ $y(t) = \tan t$ (we will see later that this is also the unique solution to this ODE.)

But $\lim_{t \to \pm(\frac{\pi}{2})} \tan t = \pm\infty$, and so $\lim_{t \to \pm\frac{\pi}{2}} y(t) = \pm\infty$.

The solution does not exist for all $t \in \mathbb{R}$...

§4.1 The theorem of existence and uniqueness for first-order nonlinear ODEs

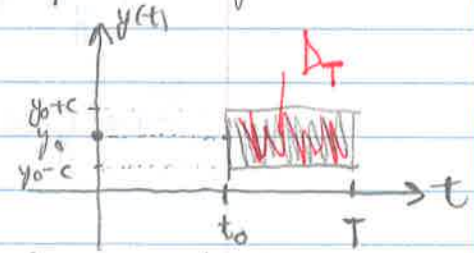
Consider the initial value problem $y'(t) = f(t, y(t))$, $y(t_0) = y_0$. $(*)$
Suppose that there exists constants

$$C, K, L > 0$$

such that the three following assumptions hold:

(A1) The function f is continuous on the slab $D_T \subset \mathbb{R}^2$ defined as

$$D_T = \{(t, y) : t \in [t_0, T] \text{ and } y \in [y_0 - C, y_0 + C]\}$$



(A2) The function $t \mapsto f(t, y_0)$ is bounded by K for every $t \in [t_0, T]$, i.e.,

$$|f(t, y_0)| \leq K \quad \forall t \in [t_0, T].$$

(A3) The function $y \mapsto f(t, y)$ is Lipschitz continuous on D_T, that is,

$$|f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2| \quad , \forall t \in [t_0, T]$$
$$\forall y_1, y_2 \in [y_0 - C, y_0 + C].$$

- Theorem (Picard - Lindelöf - 1894)

Suppose assumptions (A1), (A2), and (A3) are satisfied and that

$$C \geq \frac{K}{L}\left(e^{L(t-t_0)} - 1\right).$$

Then there exists a unique solution y to the IVP $(\underline{*})$ that is continuously differentiable on $[t_0, T]$.

Furthermore, $|y(t) - y_0| \leq C$ for every $t \in [t_0, T]$. This means $t \mapsto (t, y(t))$ belongs to D_T.

Idea

Proof: The main idea of the proof is to integrate both sides of the IVP with respect to t and apply a "fixed-point" iteration:

$$y'(t) = f(t, y(t)) \iff y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))\,ds$$

$$\iff \underbrace{y(t)} = y(t_0) + \int_{t_0}^{t} f(s, \underbrace{y(s)})\,ds.$$

y appears on both sides!

Since $y(s) = y(t_0) + \int_{t_0}^{s} f(u, y(u))\,du$, $y(t) = y(t_0) + \int_{t_0}^{t} f(s, [y(t_0) + \int_{t_0}^{s} f(u, \underbrace{y(u)})\,du])\,ds$

Can keep on substituting and hope that after enough "iterations" we "converge" somehow.

=> This motivates an iteration scheme (<u>Picard - Lindelöf "iteration"</u>)

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$$\begin{cases} y_0(t) = y_0 . \\ y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds \quad , \ n \geq 1 . \end{cases}$$

Existence • Let's use this idea to prove that a solution $t \mapsto y(t)$ to the IVP exists on $[t_0, T]$:

Write $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds$ and

$$y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds .$$

We will compute the absolute value of their difference and show that $y_n(t) \in [y_0 - c, y_0 + c]$, $\lim_{n\to\infty} y_n(t)$ exists and is continuous and in fact differentiable on $[t_0, T]$.

First ingredient • $$|y_{n+1}(t) - y_n(t)| = \left| \int_{t_0}^{t} f(s, y_n(s)) - f(s, y_{n-1}(s))\,ds \right| ,$$

$$\leq \int_{t_0}^{t} | f(s, y_n(s)) - f(s, y_{n-1}(s)) |\,ds ,$$

$$\leq L \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)|\,ds \quad \text{by assumption } \underline{(A3)} .$$

We also have that $|y_1(t) - y_0(t)| = \left| \int_{t_0}^{t} f(s, y_0)\,ds \right| \leq \int_{t_0}^{t} |f(s, y_0)|\,ds$

$$\leq \int_{t_0}^{t} K\,ds \quad \text{by assumption } \underline{(A2)}$$

$$= K(t - t_0) .$$

Combine those two together :

$$|y_2(t)-y_1(t)| \le L\int_{t_0}^{t} |y_1(s)-y_0(s)|\,ds \le \int_{t_0}^{t} kL(s-t_0)$$

$$= kL\frac{(t-t_0)^2}{2}$$

$$|y_3(t)-y_2(t)| \le L\int_{t_0}^{t} |y_2(s)-y_1(s)|\,ds \le kL^2\int_{t_0}^{t} \frac{(s-t_0)^2}{2}\,ds$$

$$= kL^2\frac{(t-t_0)^3}{2\cdot 3}$$

$\Rightarrow$ Induction: $|y_n(t)-y_{n-1}(t)| \le \frac{kL^{n-1}(t-t_0)^n}{n!}$.

Second ingredient.

Write $|y_n(t)-y_0| = \Big|(y_n(t)-y_{n-1}(t)) + (y_{n-1}(t)-y_{n-2}(t)) + (y_{n-2}(t)-y_{n-3}(t))$

$+ \dots + (y_2(t)-y_1(t)) + (y_1(t)-y_0)\Big|$.

$$\Rightarrow |y_n(t)-y_0| \le |y_n(t)-y_{n-1}(t)| + |y_{n-1}(t)-y_{n-2}(t)| + \dots + |y_1(t)-y_0|$$

$$= \sum_{j=1}^{n} |y_j(t)-y_{j-1}(t)|$$

$$\le \sum_{j=1}^{n} \frac{kL^{j-1}(t-t_0)^j}{j!} = \frac{K}{L}\left(\sum_{j=1}^{n} \frac{L^j(t-t_0)^j}{j!}\right)$$

$$= \frac{K}{L}\left(\sum_{j=0}^{n} \frac{L^j(t-t_0)^j}{j!} - 1\right)$$

$$\le \frac{K}{L}\left(\sum_{j=0}^{\infty} \frac{L^j(t-t_0)^j}{j!} - 1\right) = \underbrace{\frac{K}{L}\left(e^{L(t-t_0)} - 1\right)}_{\le C \text{ by assumption.}}$$

$\therefore$ $y_n(t) \in [y_0 - C, y_0 + C]$ and the iteration is well-defined on D_T.

Third ingredient

Next: we had $y_n(t) = y_0 + \sum_{j=1}^{n} (y_j(t) - y_{j-1}(t))$,
with $\sum_{j=1}^{n} |y_j(t)-y_{j-1}(t)| \le \sum_{j=1}^{\infty} |y_j(t)-y_{j-1}(t)| \le \frac{K}{L}(e^{L(t-t_0)}-1)$

↳ The series $\sum_{j=1}^{\infty} (y_j(t)-y_{j-1}(t))$ converges absolutely, and in fact uniformly on $t \in [t_0, T]$.

(Uniform convergence: $\forall \varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ independent of t such that for every $n > n_\varepsilon$ and all $t \in [t_0, T]$,

$$\left| \sum_{j=1}^{n} (y_j(t) - y_{j-1}(t)) - \sum_{j=1}^{\infty} (y_j(t) - y_{j-1}(t)) \right| < \varepsilon .)$$

As a consequence, $y_0 + \sum_{j=1}^{n} (y_j(t) - y_{j-1}(t))$ converges uniformly to a function $t \mapsto y(t)$ as $n \to \infty$.

$$\therefore \lim_{n\to\infty} y_n(t) = y(t) \text{ for some function } t \mapsto y(t) \text{ on } [t_0, T].$$

Furthermore, as $t \to y_n(t)$ is continuous on $[t_0, T]$ (it is produced by an integral of a continuous function), then uniform convergence implies that $t \to y(t)$ is continuous. (Fact from analysis.)

$$\therefore y(t) = \lim_{n\to\infty} y_n(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))\, ds.$$

Since $|f(s, y_n(s))| \le K$ on $[t_0, T]$, the limit can be taken inside (no proof)

$$\Rightarrow y(t) = y_0 + \int_{t_0}^{t} \lim_{n\to\infty} f(s, y_n(s))\, ds = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

by continuity of f.

Finally : By the fundamental theorem of calculus,

$$\frac{d}{dt} y(t) = \frac{d}{dt}\int_{t_0}^{t} f(s,y(s))\,ds = f(t,y(t)),$$ which is continuous.

$\Rightarrow y'(t) = f(t,y(t))$ and $t \to y'(t)$ is continuous on $[t_0,T]$!

This proves existence!

Uniqueness

Uniqueness : Assume that $t \to y(t)$ and $t \to z(t)$ are solutions to the IVP $y'(t) = f(t,y(t))$. Then for $t_0 \leq t \leq T$,

$$y(t) - z(t) = \int_{t_0}^{t} f(s,y(s)) - f(s,z(s))\,ds$$ and

$$|y(t) - z(t)| \leq \int_{t_0}^{t} |f(s,y(s)) - f(s,z(s))|\,ds$$

$$\leq L\int_{t_0}^{t} |y(s) - z(s)|\,ds$$

Letting $\omega = \max_{s\in[t_0,T]} |y(s) - z(s)|$, we find $|y(t)-z(t)| \leq L\omega(t-t_0)$.

However, if we iterate this inequality,

$$|y(t) - z(t)| \leq L\int_{t_0}^{t} |y(s)-z(s)|\,ds \leq L\int_{t_0}^{t} L\omega(s-t_0)\,ds = \omega L^2 \frac{(t-t_0)^2}{2},$$

$$|y(t) - z(t)| \leq L\int_{t_0}^{t} |y(s)-z(s)|\,ds \leq L\int_{t_0}^{t} \omega L^2 \frac{(s-t_0)^2}{2}\,ds$$

$$\leq \omega L^3 \frac{(t-t_0)^3}{2\cdot 3}$$

By induction, $|y(t) - z(t)| \leq \frac{w L^n (t-t_0)^n}{n!} \longrightarrow 0$ as $n \to \infty$.

$\therefore \; y(t) = z(t)$.

Remarks: ① If the partial derivative $\partial f/\partial y$ is continuous in a neighborhood of (t_0, y_0), then the assumptions in the theorem are satisfied as long as $T - t_0$ is sufficiently small. (Indeed this follows by the mean value theorem.) If $\frac{\partial f}{\partial y}$ exists on $[y_0 - c, y_0 + c]$, set $L = \max\limits_{(t,y) \in [t_0, T] \times [y_0 - c, y_0 + c]} \partial f/\partial y \, (t,y)$, and then we get by the MVT that $|f(t,y_1) - f(t,y_2)| \leq L |y_1 - y_2|$, $\forall \, y_1, y_2 \in [y_0 - c, y_0 + c]$.

25 July 2019 ② The Picard - Lindelöf iteration can be applied for numerical purposes. Note, however, that calculations may become long quickly (but converge fast).

Ex: We are interested in the numerical solution of the ODE

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

using the Picard iteration procedure and given $L|T - t_0| \leq \tilde{J}$. What is the sum after n iterations?

Then
$$\begin{aligned} |y(t) - y_n(t)| &= \Big| \sum_{j=n+1}^{\infty} y_j(t) - y_{j-1}(t) \Big| \leq \sum_{j=n+1}^{\infty} |y_j(t) - y_{j-1}(t)| \\ &\leq \frac{K}{L} \sum_{j=n+1}^{\infty} [L(t - t_0)]^j / j! \\ &\leq \frac{K}{L} \sum_{j=n+1}^{\infty} \tilde{J}^j / j = \frac{K}{L} \Big[e^{\tilde{J}} - \sum_{j=0}^{n} \tilde{J}^j / j! \Big] \end{aligned}$$

For $\tilde{J} = 1/10$ and $n = 10$, this gives $|y(t) - y_n(t)| \sim 2 \cdot 10^{-11}$ $\forall \, t \in [t_0, T]$.

Ex: $y'(t) = 1 + y(t)^2$, $y(0) = 0$. Then $f(t, y(t)) \equiv 1 + y(t)^2$.

① Clearly, f is continuous on any box $D_T = [t_0, T] \times [y_0 - C, y_0 + C]$.

② $|f(t, y_0)| = 1 + y_0^2 = 1 \quad (\equiv K)$.

③ $$|f(t,y_1) - f(t,y_2)| = |y_1^2 - y_2^2) = |(y_1 - y_2)(y_1 + y_2)| = |y_1 + y_2|\,|y_1 - y_2|$$
$$\leq (|y_1| + |y_2|)\,|y_1 - y_2|$$
$$\leq \underbrace{2C}_{\equiv L}\,|y_1 - y_2| \quad \text{since } D_T = [t_0, T] \times [\underbrace{y_0}_{=0} - C, \underbrace{y_0}_{=0} + C]$$

For the theorem, we require $C \geq \frac{K}{L}\left(e^{L(T - t_0)} - 1\right)$

$$\Leftrightarrow \quad 2C^2 \geq (e^{2CT} - 1)$$

$$\Leftrightarrow 0 \leq T \leq \frac{1}{2C} \log(1 + 2C^2).$$

This is satisfied if $T < 0.238$ with $C = 2$. (The estimate for T is conservative.

More importantly, existence and uniqueness cannot be guaranteed for arbitrary large T.

In fact, $y(t) = \tan t$ and $\lim\limits_{t \to \pm \pi/2} = \pm \infty$ (Finite-time blow-up.)

III/

§4.2. Nonlinear systems of ODEs

Consider a function $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ defined via $(t,\underline{y}) \to \underline{f}(t,\underline{y}(t))$, with

$$\underline{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \quad \text{and} \quad \underline{f}(t,\underline{y}(t)) = \begin{pmatrix} f_1(t,\underline{y}(t)) \\ \vdots \\ f_n(t,\underline{y}(t)) \end{pmatrix},$$

and the nonlinear system is $\underline{y}'(t) = \underline{f}(t,\underline{y}(t))$.

Ex: $y_1'(t) = y_1(t)^2 + \sin(y_2(t))$,
$y_2'(t) = t^2 e^{y_1(t)y_2(t)}$,

$$\hookrightarrow \underline{f}(t,\underline{y}(t)) = \begin{pmatrix} y_1(t)^2 + \sin(y_2(t)) \\ t^2 e^{y_1(t)y_2(t)} \end{pmatrix}.$$

Question: How do we extend the theorem of existence and uniqueness of solutions of first-order ODEs to nonlinear systems of first-order ODEs?

(This includes n^{th}-order nonlinear ODEs - remember that all such ODEs can be reduced to a system of n 1^{st}-order ODEs.)

To answer this question, we'll need some concepts from linear algebra, and specifically those of the norms of vectors and matrices.

Defn: A norm on a vector space V (finite-dimensional of dimension n) is a function

$$\|\cdot\| : V \to [0,\infty)$$

with the properties

(1) $\|\alpha \underline{v}\| = |\alpha| \cdot \|\underline{v}\|$, $\forall \alpha \in \mathbb{R}, v \in V$.
(2) $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ $\forall u, v \in V$ (triangle inequality)
(3) $\|\underline{u}\| \geq 0$, with $\|\underline{u}\| = 0$ if and only if $\underline{u} = \underline{0}$.

Ex 1: $\|\underline{u}\|_2 = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$ (Euclidean norm)

$\|\underline{u}\|_1 = |u_1| + \dots + |u_n|$ (1-norm or ℓ^1-norm)

$\|\underline{u}\|_\infty = \max\limits_{i \in \{1,\dots,n\}} |u_i|$ (∞-norm or sup-norm)

Ex 2: (Matrix norm)

Norms on $\mathbb{R}^n$ induce norms on $\mathbb{R}^{n \times n}$ as follows:

Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

Define $\|\underline{\underline{A}}\| = \sup\limits_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \dfrac{\|\underline{\underline{A}}\, \underline{v}\|}{\|v\|}$. This defines a norm on $\mathbb{R}^{n \times n}$.

In fact, $\|\underline{\underline{A}}\, \underline{v}\| = \dfrac{\|\underline{\underline{A}}\underline{v}\|}{\|\underline{v}\|} \cdot \|\underline{v}\| \leq \left(\sup\limits_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \dfrac{\|\underline{\underline{A}}\, \underline{v}\|}{\|\underline{v}\|} \right) \|\underline{v}\| = \|\underline{\underline{A}}\| \cdot \|\underline{v}\|$

$$\therefore \|\underline{\underline{A}}\, \underline{v}\| \leq \|\underline{\underline{A}}\| \cdot \|\underline{v}\|$$

We can also define norms specific to matrices without involving vectors:

$\underline{\underline{A}} \in \mathbb{R}^{m \times n}$

$\|A\|_2 = \sqrt{\text{Largest eigenvalue of } \underline{\underline{A}}^T \underline{\underline{A}}} = \left(\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$

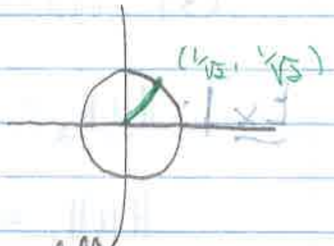
$\|\underline{\underline{A}}\|_1 = \max\limits_{1 \leq j \leq n} \left| \sum\limits_{i=1}^{n} |a_{ij}| \right|$ (Maximum absolute column of the matrix)

$\|\underline{\underline{A}}\|_\infty = \max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{n} |a_{ij}| \right)$ (Maximum absolute row of the matrix)

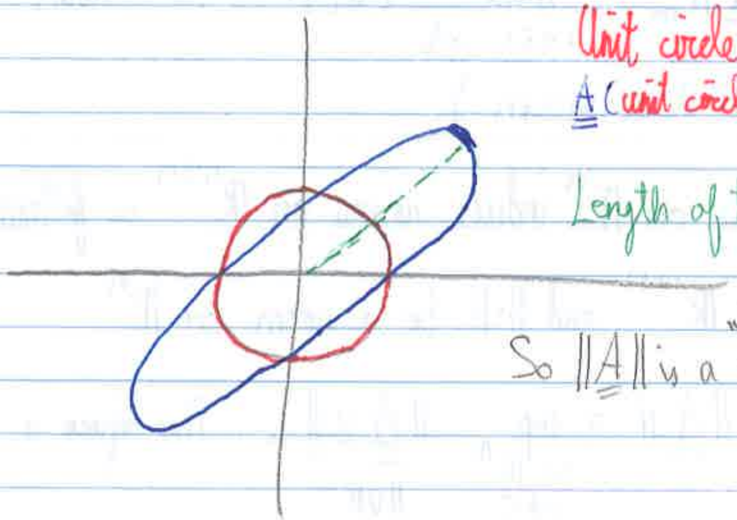
Note: If the (vector) euclidean norm measure the length of a vector, what does the induce matrix norm measure?

Ex: $V = \mathbb{R}^2$, $\underline{v} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.



The norm $\|\underline{\underline{A}}\,\underline{v}\|$ will elongate the unit circle as follows:



So $\|\underline{\underline{A}}\|$ is a "measure of elongation".

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Existence and uniqueness of nonlinear systems $y'(t) = f(t, y(t))$:

The Picard-Lindelöf theorem can be extended to the case of nonlinear systems. For the proof, the Picard iteration is the same but for vector functions $y_0, y_1, y_2, \ldots$ (component-wise).

In addition, absolute values are replaced by a vector norm $\|\cdot\|$. To estimate the integrals, we use that

$$\left\| \int_{t_0}^{t} g(s)\,ds \right\| \leq \int_{t_0}^{t} \|g(s)\|\,ds .$$
~~~

The Lipschitz continuity is defined by

$$\| \underline{f}(t,\underline{y}_1) - \underline{f}(t,\underline{y}_2) \| \leq L \| \underline{y}_1 - \underline{y}_2 \| .$$

and $\underline{y}(t_0) = \underline{y}_0$

If $\underline{f}$ is differentiable with respect to $\underline{y}$ on some region

$$D_T = \{ (t,\underline{y}) : t \in [t_0, T],\ \underline{y} \in [(y_0)_1 - c, (y_0)_1 + c] \times [(y_0)_2 - c, (y_0)_2 + c] \times \ldots \times [(y_0)_n - c, (y_0)_n + c] \}$$

then we can take $L = \max\limits_{(t,\underline{y}) \in D_T} \| \underline{\underline{J}}_{\underline{f}}(t,\underline{y}) \|$

where $\underline{\underline{J}}_{\underline{f}}(t,y)$ is called the Jacobian matrix of $\underline{f}$ and is defined via

$$\underline{\underline{J}}_{\underline{f}}(t,\underline{y}) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(t,\underline{y}) & \cdots & \frac{\partial f_1}{\partial y_n}(t,\underline{y}) \\ \vdots & & \\ \frac{\partial f_n}{\partial y_1}(t,\underline{y}) & \cdots & \frac{\partial f_n}{\partial y_n}(t,\underline{y}) \end{pmatrix}$$

($n \times n$ matrix of partial derivatives of $\underline{f}$.)

§4.3 Stability of nonlinear systems

Consider the autonomous but possibly nonlinear system $\underline{y}'(t) = \underline{f}(\underline{y})$ with $\underline{f} : \mathbb{R}^n \to \mathbb{R}^n$ (no explicit dependence on t).

Suppose that $\underline{O} \in \mathbb{R}^n$ is an equilibrium solution in that $\underline{f}(\underline{O}) = \underline{O}$ and $\underline{y}'(t) = \underline{f}(\underline{O}) = \underline{O}$ ($\underline{y}(t) \equiv \underline{O}$ is a solution).

Defn: The equilibrium solution $\underline{0} \in \mathbb{R}^n$ of $y'(t) = f(y)$ is then called Lyapunov stable if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \|y(t_0)\| \leq \delta \Rightarrow \|y(t)\| < \varepsilon, \ \forall t \in t_0.$$

(Starting "close" to $\underline{0}$ => Solution stays "close" to $\underline{0}$.)

Sometimes it is beneficial to look at a linearization of the nonlinear function f about the equilibrium solution $y(t) \equiv \underline{0}$. Here's how to do it.

Write the i^{th} component $$y_i'(t) = f_i(y_i(t)) = \underbrace{f_i(\underline{0})}_{=0} + \underline{\nabla} \cdot f_i(\underline{0}) \cdot \underline{y} + h.o.t.(|y|^2)$$

Then $$\underline{y}'(t) = \underbrace{\underline{\underline{J}}_f(\underline{0})\underline{y}}_{\text{Constant matrix}} + \underbrace{h.o.t.(|\underline{y}|^2)}_{\text{Negligible near } \underline{0} \text{ (hopefully...)}}$$

where $$\underline{\underline{J}}_f(\underline{0}) = \begin{pmatrix} \underline{\nabla} f_1(\underline{0}) \\ \vdots \\ \underline{\nabla} f_n(\underline{0}) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(\underline{0})}{\partial y_1} & \cdots & \frac{\partial f_n(\underline{0})}{\partial y_1} \\ \vdots & & \\ \frac{\partial f_1(\underline{0})}{\partial y_n} & \cdots & \frac{\partial f_n(\underline{0})}{\partial y_n} \end{pmatrix}$$

Theorem: The equilibrium solution $y(t) = \underline{0}$ of the nonlinear system $y'(t) = f(y(t))$ is Lyapunov stable if all eigenvalues of $\underline{\underline{J}}_f(\underline{0})$ satisfy

$\text{Re}\,\lambda \leq 0$ if λ is a single value
$\text{Re}\,\lambda < 0$ if λ is a multiple eigenvalue.

§5 Numerical solutions

§5.1 One-step methods

- Consider the 1D nonlinear ODE

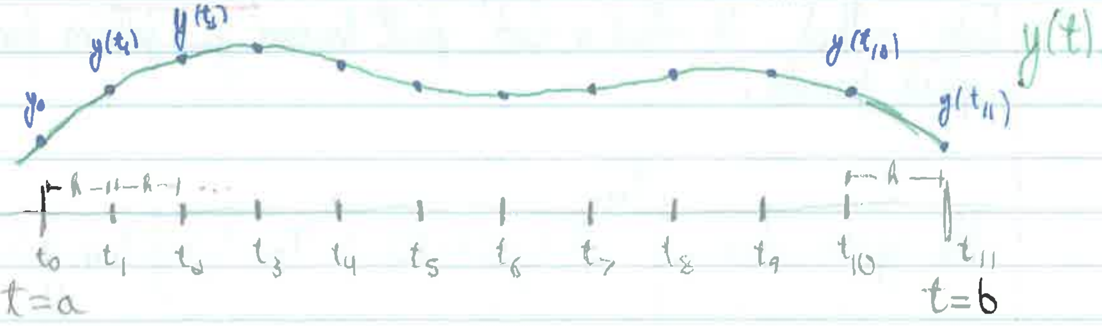
$$y'(t) = t^2 + y(t)^2 , \quad y(0)=1.$$

Suppose we want to know the value at $t = 1/2$, i.e., $y(1/2) = ?$
Unfortunately, it turns out this ODE cannot be solved in closed form! (By that, I mean $f(y(t)) = g(t)$ for some f and g.)

Most ODEs in applied problems are much more complicated than what we saw so far and they are either very hard to solve or impossible!

- Suppose we have the ODE $y'(t) = f(t, y(t))$ with $y(a) = \alpha$ for some real number a and we are interested in the value of y at $t = b > a$.

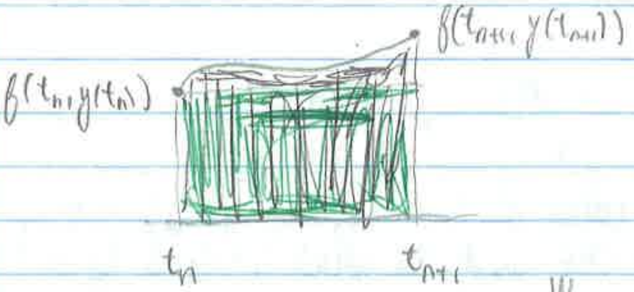
Idea 1: Divide the interval $[a,b]$ into equal subintervals with mesh points $t_n = a + nh$, $n = 0, 1, \dots, N$, where $h = \frac{b-a}{N}$ and $t_N = b$.



Numerical scheme? Integrate $y'(t) = f(t, y(t))$ with respect to t on both sides from $t = t_n$ to t_{n+1}:

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau \iff y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau .$$

Idea 2 : Approximate the integral $\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau$:



▨ $\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau$

🟩 $(t_{n+1}-t_n)\, f(t_n, y(t_n))$

$\Rightarrow$ We approximate $\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau \approx (t_{n+1}-t_n)\, f(t_n, y(t_n))$.

In that case, $y(t_{n+1}) \approx y(t_n) + \underbrace{(t_{n+1}-t_n)}_{=h} f(t_n, y(t_n))$,

and this leads to Euler's method :

$$y_{n+1} = y_n + h\, f(t_n, y_n), \quad \text{where } y_0 = y(a).$$

This is called Euler's method, or more precisely the forward Euler method or explicit Euler method. The method is called "explicit" because y_{n+1} follows from the evaluation of an explicit formula.

The approximation $\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau \approx (t_{n+1}-t_n)\, f(t_{n+1}, y(t_{n+1}))$ leads to

$$y_{n+1} = y_n + h\, f(t_{n+1}, y_{n+1}), \text{ where } y_0 = y(a).$$

This is called the backward Euler method or implicit Euler method. This method requires finding y_{n+1} (by looking for the zeros of $y_{n+1} \mapsto y_{n+1} - y_n - h\, f(t_{n+1}, y_{n+1})$) and is more complicated than the explicit Euler method. However, the backward Euler's method is often more reliable and stable (to be defined later ...) than the forward Euler's method.

- Many other approximations exist :

$$\int_{t_n}^{t_{n+1}} f(\tau,y(\tau))d\tau = \frac{1}{2}\int_{t_n}^{t_{n+1}} f(\tau,y(\tau))d\tau + \frac{1}{2}\int_{t_n}^{t_{n+1}} f(\tau,y(\tau))d\tau$$

$$\approx \frac{h}{2} f(t_n,y(t_n)) + \frac{h}{2} f(t_{n+1},y(t_{n+1}))$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(t_n,y_n) + f(t_{n+1},y_{n+1})\right]$$ (Trapezoidal method or Crank-Nicholson method.)

This is a special case of one-step-Θ-method :

$$y_{n+1} = y_n + h\left[\Theta f(t_n,y_n) + (1-\Theta) f(t_{n+1},y_{n+1})\right].$$

- $\Theta = 0$ implicit Euler method.
- $\Theta = 1/2$ Trapezoidal method.
- $\Theta = 1$ Explicit Euler method.

Defn : A numerical method is called a one-step-method if the current approximation is obtained only from the approximation at the previous time step ($y_n \Rightarrow y_{n+1}$).

The method is called explicit if y_{n+1} can be obtained explicitly from y_n and implicit otherwise.

- More generally, an explicit one-step-method has the form :

$$\begin{cases} y_{n+1} = \Phi(t_n, y_n, f(t_n,y_n)) + y_n \\ y_0 = y(a) \quad \text{(given initial value)}. \end{cases}$$

- These methods and definitions generalize to systems of ODEs as well.

Ex : $\begin{cases} y''(t) + \sin(y(t)) = 0 \end{cases}$, $y(0) = 1$,
$y'(0) = 0$.

Let $z(t) = y'(t) \Rightarrow z'(t) = y''(t) = -\sin(y(t))$.

Then $\begin{pmatrix} y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} z(t) \\ -\sin(y(t)) \end{pmatrix}$ with $\begin{pmatrix} y(0)=1 \\ z(0)=0 \end{pmatrix}$.

Numerical solution with explicit Euler scheme:

If $\underline{w}(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$ and $f(t, \underline{w}(t)) = \begin{pmatrix} z(t) \\ -\sin(y(t)) \end{pmatrix}$, then

$$\underline{w}_{n+1} = \underline{w}_n + h f(t_n, w_n)$$

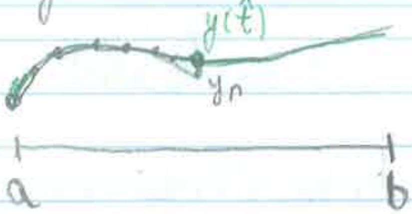
$$\Leftrightarrow \begin{pmatrix} y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ z_n \end{pmatrix} + h \begin{pmatrix} z_n \\ -\sin(y_n) \end{pmatrix}$$

You can code this in your scientific programming language of your choice.

In MATLAB :

```
T = 10.0 ;   % end time
N = 100  ;   % number of time steps
h = T/N ;
w0 = [1,0] ;  % [y(0), y'(0)]
Y  = [w0(1)];
for j =1:N
      w1 = w0 + h*[w0(2) -sin(w0(1))];
      Y = [Y w1(1)];
      w0 = w1 ;
end
plot(0:h:T, Y,'-');
```

- What can we say about these methods in the limit $\substack{h\to0\\n\to\infty}$ such that $\hat{t} \equiv nh$ remains constant?



We expect $\frac{y_{n+1}-y_n}{h} \to \frac{dy}{dt}$ in the limit $\substack{h\to0\\n\to\infty}$, and so we "expect" $y_n \to y(\hat{t})$ as $\substack{h\to0\\n\to\infty}$.

Do we have that $|y_n - y(\hat{t})| \xrightarrow[h\to0]{n\to\infty} 0$? If so, how quickly (in h)?

Defn: A one-step-method is called convergent at some point $\hat{t} \in [a,b]$ if

$$\lim_{\substack{h\to0\\n\to\infty\\ \hat{t}=nh \text{ is constant}}} \underset{\uparrow \text{ Numerical solution}}{|y_n} - \overset{}{y(\hat{t})}| = 0$$

↖ Exact solution at $\hat{t}$.

We say that the method is of order p if the error

$$|y_n - y(\hat{t})| \sim h^p \text{ as } h\to0.$$

The question is then: when do we have convergence and of which order?

Answer: $\boxed{\text{Convergence}} \longleftrightarrow \boxed{\text{Stability}} + \boxed{\text{Consistency}}$

Stability: Small perturbations in the scheme result in only small perturbations in the corresponding numerical solutions.

Consistency: The numerical solution reproduces the ODE when $h\to0$. In other words, $\lim_{h\to0}$ "Scheme" = ODE.

§5.1.1 Consistency

- Fix $\hat{t} = nh \in [a, b]$. An explicit one-step method at $\hat{t}$ is then given by

$$y_{n+1} = y_n + h\Phi(\hat{t}, y_n, f(\hat{t}, y_n)).$$

Replacing y_n with $y(\hat{t})$ and y_{n+1} with $y(\hat{t}+h)$ yields

$$y(\hat{t}+h) = y(\hat{t}) + h\,\Phi(\hat{t}, y(\hat{t}), f(\hat{t}, y(\hat{t})))$$

$$\Leftrightarrow \quad \underbrace{\frac{y(\hat{t}+h) - y(\hat{t})}{h}} = \Phi(\hat{t}, y(\hat{t}), f(\hat{t}, y(\hat{t})))$$

Taking $\begin{matrix} h \to 0 \\ n \to \infty \\ \hat{t} = nh \text{ is constant} \end{matrix}$ gives $y'(\hat{t})$, which equals $f(\hat{t}, y(\hat{t}))$ if the scheme reproduces the ODE $y'(t) = f(t, y(t))$ as $h \to 0$.

Defn: The one-step-method is called consistent if

$$\lim_{\substack{h \to 0 \\ n \to \infty \\ \hat{t} = nh \text{ is fixed}}} \Phi(\hat{t}, y(\hat{t}), f(\hat{t}, y(\hat{t}))) = f(\hat{t}, y(\hat{t})) \quad \forall \hat{t} \in [a, b].$$

Ex: For the explicit Euler method, $\Phi(\hat{t}, y(\hat{t}), f(\hat{t}, y(\hat{t}))) = f(\hat{t}, y(\hat{t}))$ by default, and so the method is consistent.

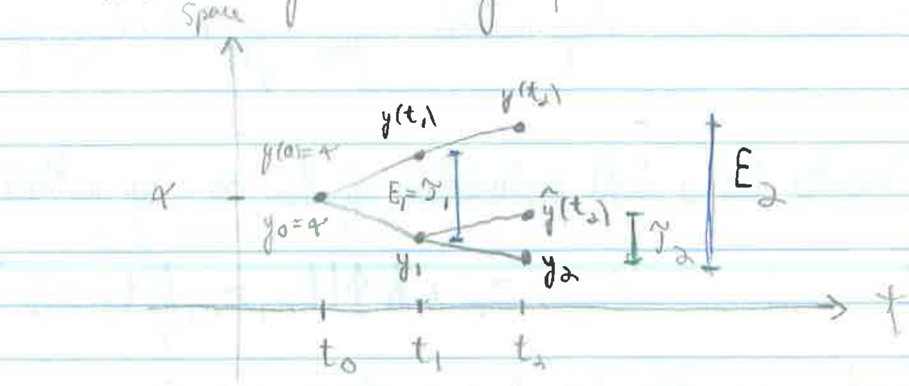
- **Defn:** We define a local truncation error (LTE) $\mathcal{T}_{n+1}(h)$ by defining a new ODE

$$\tilde{y}'(t) = f(t, \tilde{y}(t))$$
$$\tilde{y}(t_n) = y_n$$

and letting $\mathcal{T}_{n+1}(h) = \tilde{y}(t_{n+1}) - y_{n+1}$. We also let $\mathcal{T}(h) = \max\limits_{0 \le h \le N-1} \mathcal{T}_{n+1}(h)$.

We also define the global error at time t_{n+1} to be

$$E_{n+1}(h) = y(t_{n+1}) - y_{n+1}$$



So at first the LTE and global error are the same, but thereafter they are defined differently. We would like to control the global error, but the LTE is easier to measure. The one-step method is of order p if $\tau_{n+1}(h) \sim h^{p+1}$ as $h \to 0$.

Ex: What is the order of convergence of the explicit Euler's method?

$$\tau_{n+1}(h) = \tilde{y}(t_{n+1}) - y_{n+1} = \tilde{y}(t_{n+1}) - [y_n + hf(t_n, y_n)]$$
$$= \tilde{y}(t_{n+1}) - [\tilde{y}(t_n) + hf(t_n, \tilde{y}(t_n))]$$

by using $\tilde{y}(t_n) = y_n$.

Also, $\hat{y}(t_{n+1}) = \hat{y}(t_n + h)$, and we can Taylor expand this about $h=0$:

$$= \tilde{y}(t_n) + h\tilde{y}'(t_n) + \frac{h^2}{2}\hat{y}''(\xi_n), \text{ where } \xi_n \in [t_n, t_{n+1}]$$

(assume $\tilde{y}$ is twice diff.)

Since $\hat{y}'(t_n) = f(t_n, \tilde{y}(t_n))$, we find

$$\tau_{n+1}(h) = \frac{h^2}{2}\tilde{y}''(\xi_n) \implies \text{order is } p = 1.$$

§5.1.2 Stability

Consider a one-step method $y_{n+1} = y_n + h\Phi(t_n, y_n, f(t_n, y_n))$, $y_0 \equiv$ initial value of the ODE.

Consider also a slight perturbation of this one-step method:

$$z_{n+1} = z_n + h\,\Phi(t_n, z_n, f(t_n, z_n)) + \underbrace{h\,\varepsilon_n}_{\text{Perturbation}}, \quad z_0 = y_0,$$

and suppose that $|\varepsilon_n| \le \varepsilon$ for some $\varepsilon > 0$.

Stability: ε_n "small" $\Rightarrow$ $|y_n - z_n|$ "small".

Defn: A one-step method is stable if there exists a constant $C > 0$ independent of h such that $|y_n - z_n| \le C\varepsilon$ $\forall n$, $1 \le n \le N$ (h sufficiently small).

Proposition: Suppose Φ is Lipschitz continuous in the sense that $\exists L > 0$ such that

$$|\Phi(t, y, f(t,y)) - \Phi(t, z, f(t,z))| \le L|y - z|$$

then the method is stable.

Before giving the proof, let's give an example and also a result to prove the proposition:

Ex: The explicit Euler method is stable whenever the function f in $y'(t) = f(t, y(t))$ is Lipschitz continuous. Indeed, $\Phi(t, y, f(t,y)) = f(t, y(t))$ and

$$|\Phi(t, y, f(t,y)) - \Phi(t, z, f(t,z))| = |f(t,y) - f(t,z)| \le L|y - z|, \text{ where } L \equiv \text{Lipschitz constant of } f.$$

Lemma (needed for the proposition): Given two positive numbers $\alpha, \beta \in \mathbb{R}$,
(Discrete Gronwall's lemma) $\alpha > 0$, $\beta > 0$. Consider a sequence $\{\varphi_n\}_{n=1}^{\infty}$
that satisfies
- $\varphi_0 \leq \alpha$
- $\varphi_n \leq \alpha + \beta \sum_{s=0}^{n-1} \varphi_s$, $n \geq 1$,

then $\varphi_n \leq \alpha e^{n\beta}$.

Proof: Induct on n. For $n = 0$, $\alpha e^{0\beta} = \alpha = \varphi_0$. ✓

Suppose that this is true for n. Show for n+1:

$$\varphi_{n+1} \leq \alpha + \beta \sum_{s=0}^{n} \varphi_s \leq \alpha + \beta \sum_{s=0}^{n} \alpha e^{s\beta}$$
$$= \alpha \left[1 + \beta \sum_{s=0}^{n} (e^{\beta})^s \right]$$

If $a_n = \sum_{s=0}^{n} x^s$, then $x a_n - a_n = \frac{x^{n+1} - 1}{x - 1}$, and setting $x = e^{\beta}$

$$\Rightarrow \varphi_{n+1} \leq \alpha \left[1 + \beta \left(\frac{e^{(n+1)\beta} - 1}{e^{\beta} - 1} \right) \right] = \frac{\alpha}{e^{\beta-1}} \left[e^{\beta} - (1+\beta) + \beta e^{(n+1)\beta} \right]$$

Since $e^{\beta} - (1+\beta) \geq 0$, we have, as $e^{\beta(n+1)} \geq 1$,

$$\varphi_{n+1} \leq \frac{\alpha}{e^{\beta} - 1} \left[(e^{\beta} - 1 - \beta) e^{(n+1)\beta} + \beta e^{(n+1)\beta} \right] = \alpha e^{(n+1)\beta}. \checkmark$$

∎

We are now ready to prove the proposition.

Proof (proposition): Let

$$y_{j+1} = y_j + h \Phi(t_j, y_j, f(t_j, y_j)) \quad \text{with } y_0 = y(a) \text{ (initial condition)}$$
$$z_{j+1} = z_j + h \Phi(t_j, z_j, f(t_j, z_j)) + h \varepsilon_j \quad , \; z_0 = y_0.$$

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Let $z_j - y_j \equiv w_j$. Then

$$w_{j+1} = w_j + h\left[\Phi(t_j, z_j, f(t_j, z_j)) - \Phi(t_j, y_j, f(t_j, y_j))\right] + h\varepsilon_j$$

from the previous schemes in y and z. Now, write:

$$w_n = w_0 + \sum_{j=0}^{n-1} (w_{j+1} - w_j) \quad \text{(telescoping series) with } w_0 = y_0 - z_0 = 0$$

Then $|w_n| \le \sum_{j=0}^{n-1} |w_{j+1} - w_j|$

$$\le h \sum_{j=0}^{n-1} \underbrace{\left|\Phi(t_j, z_j, f(t_j, z_j)) - \Phi(t_j, y_j, f(t_j, y_j))\right|}_{\le L|z_j - y_j| \text{ by assumption}} + h \underbrace{\sum_{j=0}^{n-1} |\varepsilon_j|}_{\le \varepsilon}$$

$$\le hL\left(\sum_{j=0}^{n-1} |z_j - y_j|\right) + hn\varepsilon$$

$\Rightarrow \underbrace{|w_n|}_{\varphi_n} = \underbrace{hL}_{\beta}\left(\sum_{j=0}^{n-1} \underbrace{|w_j|}_{\equiv \varphi_j}\right) + \underbrace{hn\varepsilon}_{\equiv \psi}$. Then $\varphi_0 \le \psi$ $(w_0 = 0)$

$\varphi_n \le \psi + \beta \sum_{j=0}^{n-1} \varphi_j$

$\Rightarrow$ By the discrete Gronwall's lemma, we find

$$|w_n| \le \psi e^{n\beta} = (hn\varepsilon) e^{hnL} \quad \text{with } hn \le b-a.$$

Hence $|w_n| \le \left((b-a)e^{(b-a)L}\right)\varepsilon$ or $|w_n| \le C\varepsilon$ with $C = (b-a)e^{(b-a)L}$ independent of h.

■

§ 5.1.3 Convergence

Theorem: Convergence $\Longleftrightarrow$ Consistency + stability.

We will prove $\Longleftarrow$ only.

- Write the method $\quad y_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n))$
  y_0 = initial value of the ODE.

Let $\tilde{y}'(t) = f(t, \tilde{y}(t))$, $\tilde{y}(t_n) = y_n$. Then $\tilde{\tau}_{n+1}(h) = \hat{y}(t_{n+1}) - y_{n+1}$ (LTE)

$$= \hat{y}(t_{n+1}) - [\hat{y}(t_n) + h f(t_n, \tilde{y}(t_n))$$

Hence $\tilde{y}(t_{n+1}) = \tilde{y}(t_n) + h\,\Phi(t_n, \tilde{y}(t_n), f(t_n, \tilde{y}(t_n))) + \tilde{\tau}_{n+1}(h)$.

- By consistency, $\tilde{\tau}_{n+1}(h) \sim h^{1+p}$ as $h \to 0$ and ∞

  $\tilde{\tau}_{n+1}(h) = h\left(\tilde{\tau}_{n+1}(h) / h\right) \equiv h\,\varepsilon_{n+1}(h)$ with $|\varepsilon_{n+1}(h)|$ bounded by a number $\varepsilon(h)$ and $\varepsilon(h) \to 0$ as $h \to 0$.

- By stability, $|\hat{y}(t_{n+1}) - y_{n+1}| \le C\,\varepsilon(h) \to 0$ as $h \to 0$.

- Finally, the global error is $E_{n+1} = y(t_{n+1}) - y_{n+1}$

$$= (y(t_{n+1}) - \hat{y}(t_{n+1})) + (\hat{y}(t_{n+1}) - y_{n+1})$$

and $|E_n| \le |y(t_{n+1}) - \tilde{y}(t_{n+1})| + \underbrace{|\tilde{y}(t_{n+1}) - y_{n+1}|}_{\to 0 \text{ as } h \to \infty}$

- Now, $y(t_{n+1}) = y(t_n) + h f(t_n, y(t_n)) + O(h^2)$ (Taylor expansion)

  $\hat{y}(t_{n+1}) = \hat{y}(t_n) + h f(t_n, \hat{y}(t_n)) + O(h^2)$ (Taylor expansion) and $\tilde{y}(t_n) = y_n$. So

$$|y(t_{n+1})-\hat{y}(t_n)| \le |y(t_n)-\hat{y}(t_n)| + \underbrace{h\,|f(t_n,y(t_n))-f(t_n,\tilde{y}(t_n))| + |O(h^2)|}_{\to 0 \text{ as } h \to 0}$$

$$\hookrightarrow \le |y(t_n)-\hat{y}(t_n)| \quad \text{as } h \to 0$$

A similar argument yields $|y(t_n)-\hat{y}(t_n)| \le |y(t_{n-1})-\hat{y}(t_{n-1})|$ as $h \to 0$

By induction, $\lim_{h \to 0} |y(t_{n+1})-\hat{y}(t_{n+1})| \le |y(t_0)-\hat{y}(t_0)| = |y_0-y_0| = 0$.

$$\therefore \lim_{h\to 0} E_{n+1}(h) = 0 .$$

■

Ex: The Euler explicit method is stable if f is Lipschitz constant, and it is consistent of order $p = 1$.

As a result, this method is convergent (of order $p=1$), meaning $\exists C>0$ s.t.

$$|y(t_n) - y_n| \le Ch \quad (hn = t_n).$$

Ex: Trapezoidal method & Henn's method

$$y_{n+1} = y_n + \frac{h}{2}\Big[f(t_n,y_n) + f(t_n,\underbrace{y_{n+1}}_{\approx\, y_n + hf(t_n,y_n)})\Big] \quad \text{(approximate)}$$

Henn's method: $g_{n+1} = y_n + h\cdot\underbrace{\frac{1}{2}\Big[f(t_n,y_n) + f(t_{n+1}, y_n + f(t_n,y_n))\Big]}_{=\Phi(}$ (Henn's method)

Let's verify that Heun's method is stable and consistent:

Stability: Suppose f is Lipschitz in y. Then

$$|\Phi(t,y,f(t,y))-\Phi(t,z,f(t,z))| = \frac{1}{2}\Big|f(t,y)+f(t+h,\, y+f(t,y))-f(t,z)$$
$$-f(t+h,\, z+f(t,z))\Big|$$

$$\leq \frac{1}{2}|f(t,y)-f(t,z)| + \frac{1}{2}|f(t+h,y+hf(t,y))-f(t+h,z+hf(t,z))|$$
$$\leq \frac{L}{2}|y-z| + \frac{L}{2}|y+hf(t,y)-z-hf(t,z)|$$
$$\leq \frac{L}{2}|y-z| + \frac{L}{2}|y-z| + \frac{Lh}{2}|f(t,y)-f(t,z)|$$
$$\leq L|y-z| + \frac{L^2h}{2}|y-z| = \left(L+\frac{L^2h}{2}\right)|y-z| \leq \left(L+\frac{L^2(b-a)}{2}\right)|y-z|.$$

$\Rightarrow$ The method is stable.

Consistency: We must compute $\lim\limits_{\substack{h\to 0\\ n\to\infty\\ t=nh \text{ is constant}}} \Phi(t,y,f(t,y))$.

Then $\lim\limits_{h\to 0}\left[\Phi(t,y,f(t,y))\right] = \lim\limits_{h\to 0}\left[\frac{1}{2}(f(t,y)+f(t+h,\, y+hf(t,y)))\right]$

$$= f(t,y) \qquad \checkmark \text{ consistent.}$$

$\therefore$ The method is convergent!

What is the order of method? Let $\tau_{n+1}(h) = \tilde{y}(t_{n+1})-y_{n+1}$, $\quad \tilde{y}'(t)=f(t,y)$, $\tilde{y}(t_n)=y_n$

$\hookrightarrow$ f is three times differentiable

$$\tau_{n+1}(h) = \tilde{y}(t_{n+1}) - \left[y_n + \frac{h}{2}\Big(f(t_n,y_n)+f(t_{n+1},\, y_n+hf(t_n,y_n))\Big)\right]$$

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- Taylor expand $\hat{y}(t_{n+1}) = \hat{y}(t_n + h)$ about $h = 0$:

$$\tilde{y}(t_{n+1}) = \tilde{y}(t_n) + h\tilde{y}'(t_n) + \frac{h^2}{2}\tilde{y}''(t_n) + O(h^3)$$

$$= y_n + hf(t_n, y_n) + \frac{h^2}{2}\tilde{y}''(t_n) + O(h^3)$$

Hence $\mathcal{T}_{n+1}(h) = \frac{h}{2}f(t_n, y_n) + \frac{h^2}{2}\tilde{y}''(t_n) - \frac{h}{2}f(t_{n+1}, y_n + hf(t_n, y_n)) + O(h^3)$.

- As $t_{n+1} = t_n + h$, we can write $f(t_{n+1}, y_n + hf(t_n, y_n)) = f(t_n + h, y_n + hf(t_n, y_n))$.

Taylor expand the first argument about $h = 0$:

$$f(t_n + h, \circledast) = f(t_n, (*)) + hf'(t_n, (*)) + O(h^2)$$

$$\Rightarrow f(t_{n+1}, y_n + hf(t_n, y_n)) = f(t_n, y_n + hf(t_n, y_n)) + h\frac{\partial f}{\partial t}(t_n, y_n + hf(t_n, y_n)) + O(h^2).$$

- Taylor expand $f(t_n, y_n + hf(t_n, y_n))$ and $\frac{\partial f}{\partial t}(t_n, y_n + hf(t_n, y_n))$ about $h = 0$:

$$f(t_n, y_n + hf(t_n, y_n)) = f(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(y_n, t_n) + O(h^2)$$

$$f'(t_n, y_n + hf(t_n, y_n)) = \frac{\partial f}{\partial t}(t_n, y_n) + O(h).$$

- Taken together:

$$f(t_n + h, y_n + hf(t_n, y_n)) = f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + h\frac{\partial f}{\partial y}(t_n, y_n)f(t_n, y_n)$$

Hence $\mathcal{T}_{n+1}(h) = \frac{h}{2}f(t_n, y_n) + \frac{h^2}{2}\tilde{y}''(t_n) - \frac{h}{2}[f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + h\frac{\partial f}{\partial y}(t_n, y_n)f(t_n, y_n)] + O(h^3)$

$=0$

$$\Rightarrow \tau_{n+1}(h) = \frac{h^2}{2}\left[\frac{d^2}{dt^2}[\tilde{y}(t)]\Big|_{t=t_n} - \frac{\partial f}{\partial t}(y_n,t_n) - \frac{\partial f}{\partial y}(y_n,t_n)\, f(y_n,t_n)\right] + O(h^3)$$

But $\frac{d}{dt}\left(\frac{d}{dt}\hat{y}(t)\right) = \frac{d}{dt}\left(f(t,\hat{y}(t))\right) = \frac{\partial f}{\partial t}(y_n,t_n) + \frac{\partial f}{\partial y}(y_n,t_n)\cdot f(y_n,t_n)$ by the multivariable chainrule.

$\Rightarrow \tau_{n+1}(h) = O(h^3)$ and the consistency is of order 2.

§5.1.4 Runge-Kutta methods

Idea: Compute the integral $\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau$ using intermediate values between t_n and t_{n+1}.

Ex: $y_{n+1} = y_n + h\left[a f(t_n,y_n) + b f(t_n+\alpha h,\ y_n+\beta h f(t_n,y_n))\right]$

- The method is consistent if and only if $a+b=1$.
- The order of the method is 2 $(\tau_{n+1}(h) \sim O(h^3))$ if and only if $\alpha=\beta$ and $a = 1-1/2\alpha$, $b = 1/2\alpha$ $(\alpha \neq 0)$.

1) $\alpha=\beta=1/2$, $a=0$, $b=1$.

$y_{n+1} = y_n + h f\left(t_n+\frac{h}{2},\ y_n+\frac{h}{2} f(t_n,y_n)\right)$. (Modified Euler's method)

2) $\alpha=\beta=1$, $a=b=1/2$.

$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n,y_n) + f(t_n+h,\ y_n+hf(t_n,y_n))\right]$ (Heun's method)

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Def^n: The classical Runge - Kutta method of 4^{th} order :

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 3k_3 + 4k_4 \right]$$

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + h/2, y_n + h/2\, k_1)$$
$$k_3 = f(t_n + h/2, y_n + h/2\, k_2)$$
$$k_4 = f(t_n + h \quad , y_n + h \quad)$$

The order of convergence is $p = 4$, meaning the local truncation error $T_{n+1}(h) \sim O(h^5)$, which is very accurate for small h. In fact, the global error is $\sim O(h^4)$.

The Runge - Kutta method of 4^{th}-order offers a nice balance between numerical computations (not too many) and accuracy and is widely used in the applied sciences and engineering.