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Nonlinear dynamics of finite-size particles in unsteady fluid flows, and spatially linear solutions to the Navier–Stokes equations

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Nothing in the world can take the place of persistence. Talent will not; nothing is more common than unsuccessful men with talent. Genius will not; unrewarded genius is almost a proverb. Education will not; the world is full of educated derelicts. Persistence and determination alone are omnipotent. The slogan ″press on″ has solved and always will solve the problems of the human race.

Calvin Coolidge

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Abstract

This thesis presents the results of two research projects on two different problems of unsteady fluid dynamics.

The first project is on the dynamics of inertial particles in fluids. Recent experimental and numerical observations have shown the significance of the Basset–Boussinesq memory term in the Maxey–Riley equation for modeling the dynamics of small spherical rigid bodies (or inertial particles) in Newtonian fluids. These observations suggest an algebraic decay to an asymptotic state, as opposed to the exponential convergence in the absence of the memory term. Here, we prove that the observed algebraic decay is a universal property of the Maxey–Riley equation. Specifically, the particle velocity decays algebraically to a limit that is $\mathcal{O}(\epsilon)$-close to the fluid velocity, where $0 < \epsilon \ll 1$ is proportional to the square of the ratio of the particle radius over the fluid characteristic length-scale. These results follows from a sharp analytic upper bound that we derive for the particle velocity. We also present a first proof of the global existence and uniqueness of mild solutions to the Maxey–Riley equation. Finally, we present by means of a formal series expansion a reduced-order equation for the asymptotic dynamics of the Maxey-Riley equation.

The second project is on the class of spatially linear unsteady solutions to the Navier-Stokes equations. We show that a smooth linear unsteady velocity field $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ solves the incompressible Navier–Stokes equation if and only if the matrix $\mathbf{A}(t)$ has zero trace, and $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ is symmetric. In two dimensions, these constraints imply that $\mathbf{A}(t)$ is the sum of an arbitrary time-dependent traceless symmetric matrix and an arbitrary constant skew-symmetric matrix. Thus one can verify by inspection if an unsteady spatially linear vector field is a Navier–Stokes solution. In three dimensions, we obtain a simple ordinary differential equation that $\mathbf{A}(t)$ must solve. The formulas enable the construction of simple yet unsteady and dynamically consistent flows for testing numerical schemes and verifying coherent structure criteria.

Zusammenfassung

Die folgende Arbeit präsentiert Ergebnisse aus zwei verschiedenen Projekten zur Dynamik instationärer Strömungen.

Das erste Projekt untersucht die Dynamik von Partikeln in Fluiden. Neue Experimente und numerische Beobachtungen haben die Bedeutung des Basset-Boussinesq-Gedächtnisterms in der Maxey-Riley-Gleichung für die Modellierung der Dynamik kleiner sphärischer Starrkörper (Partikel) in Newton'schen Flüssigkeiten gezeigt. Diese Beobachtungen legen einen algebraische Dämpfung in einen asymptotischen Zustand nahe, im Gegensatz zur exponentiellen Konvergenz in Abwesenheit des Gedächtnisterms. Hier zeigen wir, dass der beobachtete algebraische Abfall eine universale Eigenschaft der Maxey-Riley-Gleichung ist. Insbesondere fällt die Partikel-Geschwindigkeit algebraisch auf einen Grenzwert, der $\mathcal{O}(\epsilon)$-nah an der Strömungsgeschwindigkeit liegt, wobei $0 < \epsilon \ll 1$ proportional zum Quadrat des Verhältnisses aus Partikelradius zur charakterischen Längenskala des Fluids ist. Diese Ergebnisse folgen aus einer analytischen, scharfen oberen Schranke, die wir für die Partikelgeschwindigkeit herleiten. Wir zeigen ebenfalls eine ersten Beweis der globalen Existenz und Eindeutigkeit von schwachen Lösungen der Maxey-Riley-Gleichung. Abschliessend präsentieren wir eine formale Reihenentwicklung, und daraus folgend, eine reduzierte Gleichung für die asymptotische Dynamik der Maxey-Riley-Gleichung.

Das zweite Projekt behandelt die Klasse räumlich-linearer, instationärer Lösungen der Navier-Stokes-Gleichungen. Wir zeigen, dass ein glattes, lineares und zeitabhängiges Geschwindigkeitsfeld $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ die inkompressiblen Navier-Stokes-Gleichungen genau dann löst, wenn die Spur der Matrix $\mathbf{A}(t)$ verschwindet, und $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ eine symmetrische Matrix ist. In zwei Dimensionen bedeuten diese Bedingungen, dass $\mathbf{A}(t)$ die Summe aus einer beliebigen, zeitabhängigen und spurfreien Matrix und einer beliebigen konstanten schiefsymmetrischen Matrix ist. Man kann hiermit also direkt prüfen, ob ein zeitabhängiges, räumlich-lineares Vektorfeld die Navier-Stokes-Gleichungen löst. In drei Dimensionen erhalten wir eine unkomplizierte gewöhnliche Differentialgleichung, die $\mathbf{A}(t)$ erfüllen muss. Diese Formeln erlauben die Konstruktion einfacher, aber gleichzeitig zeitabhängiger und dynamisch konsistenter Flüsse zur Erprobung numerischer Schemata und zur Bestätigung von Kriterien für kohärente Strukturen.

Résumé

Ce mémoire présente les résultats de deux projets de recherches portant sur deux différent problèmes de la dynamique des fluides non permanents.

Le premier projet porte sur la dynamique des particules inertielles dans les fluides. Des observations expérimentales et numériques récentes ont démontré la pertinence du terme de mémoire de Basset–Boussinesq dans l'équation de Maxey–Riley pour modéliser la dynamique d'un corps rigide sphérique (ou particule inertielle) dans un fluide Newtonien. Ces observations suggèrent une convergence algébrique à un état asymptotique, en contraste à une convergence exponentielle en l'absence du terme de mémoire. Nous prouvons que cette convergence algébrique est une propriété universelle de l'équation de Maxey–Riley. Nous démontrons que la vitesse d'une particule inertielle converge de manière algébrique à une limite qui est $\mathcal{O}(\epsilon)$-proche de la vitesse du fluide ambiant, où $0 < \epsilon \ll 1$ est proportionnel au carré du ratio du rayon de la particule sur la longueur caractéristique du fluide. Ces résultats suivent d'une borne supérieure analytique que nous dérivons pour la vitesse de la particule. Nous présentons aussi une première preuve de l'existence et unicité globale des solutions allégées de l'équation de Maxey–Riley. Enfin, nous présentons un calcul formel d'une équation à dimension réduite décrivant la dynamique asymptotique de l'équation de Maxey–Riley.

Le second projet porte sur les solutions spatialement linéaires non permanents des équations de Navier-Stokes. Nous démontrons que le champ de vitesse linéaire et lisse $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ résout l'équation de Navier–Stokes si et seulement si la matrice $\mathbf{A}(t)$ a une trace nulle et si $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ est symétrique. En deux dimensions, ces contraintes impliquent que $\mathbf{A}(t)$ est la somme d'une matrice symétrique sans trace qui peut dépendre du temps et d'une matrice antisymétrique constante. Il est donc possible avec ces résultats de vérifier si un champ vectoriel spatialement linéaire et non permanent résout les équations de Navier-Stokes. En trois dimensions, nous obtenons une équation différentielle ordinaire simple que la matrice $\mathbf{A}(t)$ doit résoudre. Ces formules rendent donc possible la construction d'écoulements de Navier–Stokes simples, mais non permanents pour tester des méthodes numériques et vérifier des critères de structures cohérentes.

Sommario

In questo lavoro di tesi vengono presentati i risultati di due progetti inerenti sistemi fluidodinamici non stazionari.

Il primo progetto riguarda la dinamica di particelle inerziali immerse in un fluido. Recenti risultati sperimentali, insieme ad osservazioni numeriche, hanno mostrato la rilevanza dei termini di memoria di Basset–Boussinesq nelle equazioni di Maxey–Riley che governano la dinamica di piccole sfere rigide (particelle inerziali) immerse in fluidi Newtoniani. Queste osservazioni suggeriscono una convergenza algebrica ad uno stato asintotico anzichè una convergenza esponenziale come nel caso in assenza dei termini di memoria. In questo lavoro, dimostriamo che la convergenza algebrica osservata è una proprietà universale delle equazioni di Maxey–Riley. In particolare, la velocità delle particelle converge algebricamente ad un limite che è $\mathcal{O}(\epsilon)$-vicino alla velocità del fluido, dove $0 < \epsilon \ll 1$ è proporzionale al quadrato del rapporto tra il raggio della particella e la lunghezza caratteristica del fluido. Questi risultati derivano da un limite superiore, qui derivato, per la velocità della particella. Presentiamo anche una prima dimostrazione dell'esistenza e unicità della forma debole della soluzione delle equazioni di Maxey–Riley. Infine, presentiamo tramite un'espansione in serie, la forma ridotta della dinamica asintotica delle equazioni di Maxey–Riley.

Il secondo progetto riguarda la classe di soluzioni delle equazioni di Navier–Stokes non stazionarie e lineari nello spazio. Dimostriamo che un campo di velocità differenziabile $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ è soluzione delle equazioni Navier–Stokes incomprimibili se e solo se $\mathbf{A}(t)$ ha traccia zero, e $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ è simmetrica. In due dimensioni, questi vincoli implicano che $\mathbf{A}(t)$ è la somma di una matrice non stazionaria simmetrica con traccia nulla e una matrice antisimmetrica stazionaria. Questo risultato permette di verificare se un campo di velocità non stazionario e lineare nello spazio è soluzione delle equazioni di Navier–Stokes. In tre dimensioni, invece, otteniamo una semplice equazione differenziale matriciale che $\mathbf{A}(t)$ deve risolvere. Questi risultati permettono di costruire semplici soluzioni delle equazioni di Navier–Stokes e permettono, quindi, di testare schemi numerici per l'identificatione di strutture coerenti.

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Chapter 1

General introduction

This thesis presents the results of two research projects that I completed during my master studies under the supervision of Prof. George Haller, my mentor at the ETH Zürich. The first of the two projects was done in collaboration with Dr. Mohammad Farzmand, a former doctoral student of Prof. George Haller. The two projects stemmed from problems from two different areas of unsteady fluid dynamics.

For the first project, we proved global existence, uniqueness, and asymptotic properties of solutions to the Maxey–Riley equation, a widely accepted model for describing the motion of small spherical rigid bodies immersed in fluids [1]. We wrote our results in a paper [2], which we submitted to the *Journal of Nonlinear Science* in September 2014. It was accepted for publication in April 2015 and published in May 2015.

For the second project, we derived the explicit form of spatially linear solutions to the Navier–Stokes equations. We showed how these spatially linear solutions are relevant for verifying vortex and coherent structure criteria. We are currently working on a derivation of the explicit form of higher-order polynomial solutions to the Navier–Stokes equations and will publish our results elsewhere.

I want to give in this general introduction a brief account of how I became involved in these projects.

1.1 Presentation of the subject

In the fall of 2013 I began my master studies in applied mathematics and joined the nonlinear dynamics group of Prof. George Haller at the ETH Zürich. At the first group meeting I met the members of the group, including Mohammad Farazmand, who gave an update on his research. He told us about a problem that involved the differential equation

$$\begin{aligned} &\frac{\mathrm{d}\psi}{\mathrm{d}t} + \kappa \frac{\mathrm{d}^{1/2}\psi}{\mathrm{d}t^{1/2}} + \psi = 0, \\ &\psi(0) = 1, \end{aligned} \tag{1.1}$$

where $\psi(t)$ is a function of time t, κ is some positive constant, and $\mathrm{d}^{1/2}\psi/\mathrm{d}t^{1/2}$ is the half-derivative (technically, the Riemann–Liouville fractional derivative of order $1/2$), which is defined as

$$\frac{\mathrm{d}^{1/2}\psi}{\mathrm{d}t^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_0^t \frac{\psi(s)}{\sqrt{t-s}}\,\mathrm{d}s\right).$$

Farazmand wanted to know the solution to (1.1) and its behavior in the limit $t \to \infty$.

He wanted to know the solution because, as he elaborated, the differential equation appeared in the Maxey–Riley equation, a widely accepted model for describing the motion of small spherical rigid bodies (inertial particles) immersed in fluids [1]. In the context of the Maxey–Riley equation, the differential equation (1.1) describes the relative velocity of an infinitesimal inertial particle with respect to the ambient fluid flow. To understand the long-term dynamics of such particles, therefore, it was necessary to find the solution to (1.1) and its asymptotic behavior. In spite of the widespread use of the Maxey–Riley equation in applications, the analysis of its solutions remained an open problem. Farazmand and Haller [3] had studied some of its basic properties, but the asymptotic analysis of the Maxey–Riley equation was challenging and still had to be done.

No one in the group knew the answer; I didn't even know fractional derivatives existed. It was obvious that without the fractional derivative the solution was $\psi(t) = e^{-t}$, an exponential decaying to zero in time. Farazmand suggested, based on his experience and preliminary literature review, that (1.1) could be solved via a Laplace transform, a mathematical tool I was well acquainted with.

I have always liked to solve mathematical problems involving extensive calculations, and so after the group meeting I decided I would try my hand at finding the solution. I spent several hours a day at the ETH-Bibliothek to solve the problem. After weeks of laborious work I finally found and wrote down the solution to (1.1) in terms of elementary functions. I also managed to show by means of an asymptotic series expansion that the solution decayed to zero in time algebraically as $\mathcal{O}\left(t^{-3/2}\right)$. I showed my solution to Farazmand, who was pleased with the result.

This was the starting point of my collaboration with Farazmand and Haller on the Maxey–Riley equation. Later in the following months we used the solution that I found to prove global existence and uniqueness, upper bounds, and asymptotic properties of solutions to the Maxey–Riley equation, effectively completing the work he and Haller had begun [3]. We wrote our results in a manuscript and submitted it to the *Journal of Nonlinear Science* in September 2014. The paper was accepted for publication in April 2015 and published in May 2015.

In November 2014 Haller mentioned during coffee break a new problem he was working on. It was about finding the explicit form of matrices $\mathbf{A}(t)$ and vectors $\mathbf{f}(t)$ such that the linear velocity field

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \tag{1.2}$$

solves the Navier–Stokes equation. The problem had applications to verifying coherent structures criteria in unsteady fluid dynamics, a subject important to him. But he was too busy with his other research projects and his teaching responsibilities to give enough time to the problem. I proposed to collaborate and solve the problem for him.

I worked on the problem whenever I was not busy with coursework. It was less technical than the previous project, but I could not find much on the subject in the fluid dynamics literature. I found general existence conditions for the solutions to (1.2) [4, 5], but that was all. Nevertheless I eventually derived necessary and sufficient conditions for the matrices $\mathbf{A}(t)$ and vectors $\mathbf{f}(t)$ in (1.2) to solve the Navier–Stokes equations. I found from these conditions the explicit form of such two- and three-dimensional matrices $\mathbf{A}(t)$, and other properties of spatially linear velocity field solutions (1.2) to the Navier–Stokes equations. (Later Haller and I found that these necessary and sufficient conditions had been derived before by Craik and Criminale [6]. The results on the explicit form of matrices $\mathbf{A}(t)$ that I found, however, was novel.) At the present time I am working on deriving the explicit form of spatially higher-order polynomial solutions to the Navier-Stokes equations.

1.2 Organization of the thesis

Chapter 2, with the exception of section 2.5, follows closely a paper [2] that I wrote and published as first author in collaboration with Dr. Mohammad Farazmand and Prof. George Haller. It presents global existence and uniqueness, upper bounds, and asymptotic properties of solutions to the Maxey–Riley equation. The contents of section 2.5 is novel, and present a formal calculation of a reduced-order equation for the asymptotic dynamics of the Maxey–Riley equation.

Chapter 3 is a collection of results that I researched and wrote in collaboration with Prof. George Haller. It presents necessary and sufficient conditions for the linear unsteady velocity field $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ to solve the Navier-Stokes equations, the explicit form of such matrices $\mathbf{A}(t)$, and the application of such unsteady velocity fields for verifying coherent structures criteria. It follows closely the material that I submitted as a semester paper project for credit in August 2015.

Chapter 2

Asymptotic dynamics of inertial particles with memory

2.1 Introduction

The motion of a solid body transported by an ambient Newtonian fluid flow can, in principle, be determined by solving the Navier–Stokes equations with appropriate moving boundary conditions [7, 8]. The resulting partial differential equations are, however, too complicated for mathematical analysis. Their numerical solutions are computationally expensive and yield little insight.

For the motion of a small spherical rigid body (or inertial particle), however, one can derive a reliable model by accounting for all the forces exerted on the particle due to the solid-fluid interaction. Stokes [9] made the first attempt to obtain such a model for the oscillatory motion of an inertial particle. Later, Basset [10], Boussinesq [11], and Oseen [12] studied the settling of a solid sphere under gravity in a fluid at rest. The resulting equation is known as the BBO equation. To study the motion of inertial particles in a non-uniform unsteady flow, Tchen [13] wrote the BBO equation in a frame of reference moving with the fluid, accounting for various inertial forces that arise in this frame.

The exact form of the forces exerted on the particle has been debated and corrected by several authors (e.g., Corrsin and Lumley [14]). A widely accepted form of the forces was derived by Maxey and Riley [1] from first principles. The resulting equation, with the later correction of Auton et al. [15] to the added mass term, is usually referred to as the Maxey–Riley (MR) equation.

To describe the MR equation, let $\mathbf{u} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^n$ denote a known velocity field describing the flow of a fluid in an open spatial domain $\mathcal{D} \subseteq \mathbb{R}^n$. Here, $n = 2$ or $n = 3$ for two- and three-dimensional flows, respectively. A fluid trajectory is then the solution of the differential equation $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$ with some initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. An inertial particle, however, follows a different trajectory $\mathbf{y}(t) \in \mathcal{D}$. The particle velocity $\mathbf{v}(t) = \dot{\mathbf{y}}(t)$ satisfies the Maxey–Riley equation

$$\begin{aligned}
\rho_p \dot{\mathbf{v}} = \rho_f \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} & \qquad \text{(Force exerted by the undisturbed fluid)} \\
+ (\rho_p - \rho_f)\mathbf{g} & \qquad \text{(Buoyancy force)} \\
- \frac{9\nu\rho_f}{2a^2}\left(\mathbf{v} - \mathbf{u} - \frac{a^2}{6}\Delta\mathbf{u}\right) & \qquad \text{(Stokes drag)} \\
- \frac{\rho_f}{2}\left[\dot{\mathbf{v}} - \frac{\mathrm{D}}{\mathrm{D}t}\left(\mathbf{u} + \frac{a^2}{10}\Delta\mathbf{u}\right)\right] & \qquad \text{(Added mass term)} \\
- \frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\left[\int_{t_0}^{t} \frac{\dot{\mathbf{w}}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{\mathbf{w}(t_0)}{\sqrt{t-t_0}}\right], & \qquad \text{(Basset–Boussinesq memory term)}
\end{aligned} \tag{2.1}$$

where

$$\mathbf{w}(t) = \mathbf{v}(t) - \mathbf{u}(\mathbf{y}(t), t) - \frac{a^2}{6}\Delta\mathbf{u}(\mathbf{y}(t), t).$$

Here, ρ_p and ρ_f are, respectively, the particle and fluid densities, ν is the kinematic viscosity of the fluid, a is the particle radius, and $\mathbf{g}$ is the constant gravitational acceleration vector. The initial conditions for the inertial particle are given as $\mathbf{y}(t_0) = \mathbf{y}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$, for some $t_0 \in \mathbb{R}$. The material derivative $\frac{\mathrm{D}}{\mathrm{D}t} := \partial_t + \mathbf{u} \cdot \nabla$ denotes the time derivative along a fluid trajectory.

The right-hand side in (2.1) contains the various forces exerted on the particle. These forces have varying orders of magnitude. In particular, the Basset–Boussinesq memory term, accounting for the lagging boundary layer developed around the sphere, is routinely neglected on the grounds that it is insignificant compared to the Stokes drag and added mass [16, 17]. Recent experimental and numerical studies, however, show that the memory term influences the dynamics of inertial particles significantly, and hence cannot be generally neglected [18–23]. This is the case even for heavy particles, for which the memory term becomes very small [21].

The numerical simulations of Daitche and Tél [21] and Guseva et al. [22], in particular, show the position of the particle to converge to its asymptotic limit algebraically. This is fundamentally different from the exponential convergence arising in the absence of the memory term [24–26]. Here, we prove that the observations of Daitche and Tél, and Guseva and al. [21, 22] are a universal property of the MR equation with memory, irrespective of the fluid flow carrying the particles.

The MR equation was originally derived under the assumption that $\mathbf{w}(t_0) = \mathbf{0}$. Later, Maxey [27] modified the original formulation to lift this unphysical restriction, obtaining equation (2.1) above. This equation can be written as a system of nonlinear fractional differential equations [3, 28] in terms of the particle position $\mathbf{y}$ and relative velocity $\mathbf{w}$. Although there exist fundamental results for special classes of fractional differential equations (e.g., [29]), the MR equation does not fit in any of these classes and requires separate treatment.

Even the existence and uniqueness of solutions to the MR equation is unclear. Only recently have Farazmand and Haller [3] proved *local* existence and uniqueness of its mild solutions. They also showed that the MR equation admits strong solutions only under the unphysical assumption $\mathbf{w}(t_0) = 0$. Here, we prove *global* existence and uniqueness of mild solutions to the MR equation for any initial condition $\mathbf{w}(t_0)$.

Understanding the asymptotic dynamics of inertial particles is of interest in several environmental problems, such as the clustering of garbage patches in the ocean [30], the transport of airborne pollutants in the atmosphere [31], and the processes of rain initiation in clouds [32]. To this end, Haller and Sapsis [26] derived a reduced-order equation for the asymptotic motion of the Maxey–Riley equation without the Basset-Boussinesq memory term. The reduced-order equation has half the dimension of the MR equation, a fact simplifying both the qualitative analysis and numerical simulations of inertial particles dynamics. This equation describes the dynamics on a globally attracting slow manifold of the MR equation without memory that Haller and Sapsis showed to exist. It is unclear, however, that the MR equation, with memory, has a globally attracting slow manifold and hence that a reduced-order equation for its asymptotic dynamics can be derived rigorously. Here, we present a formal calculation of a reduced-order equation for the asymptotic dynamics of the MR equation with memory. This reduced-order equation is justified provided the MR equation with memory has a globally attracting slow manifold, which we do not attempt to prove.

We start by rewriting the MR equation in dimensionless form as a system of nonlinear fractional differential equations in terms of the particle position $\mathbf{y}$ and the function $\mathbf{w}$. After rescaling time, we compute the solution to the MR equation in the limit of infinitesimally small particles, and then obtain integral equations for the MR equation for arbitrary particle sizes. We then use these integral equations to prove an analytic upper bound for the velocity $\mathbf{v}$ of a small particle of radius a. Next, we show that $\mathbf{v}$ decays algebraically to an asymptotic state that is $\mathcal{O}(\frac{a^2}{L^2})$-close to the fluid velocity $\mathbf{u}$, where L is a characteristic length scale of the fluid flow. We demonstrate these properties numerically on the double gyre flow model of Shadden et al. [33]. We then construct a specific continuation method to prove global existence and uniqueness of mild solutions for the MR equation. Finally, we derive by means of a formal series expansion a reduced-order equation for the asymptotic dynamics of the MR equation with memory.

2.2 Preliminaries

2.2.1 The Maxey–Riley equation in dimensionless form

We rewrite the Maxey–Riley equation (2.1) in a form more appropriate for mathematical analysis. First, we rescale space, velocities, and time using the characteristic length scale L, the characteristic velocity U, and the characteristic time scale $T = L/U$. Using the resulting dimensionless variables $\mathbf{y} \mapsto \mathbf{y}/L$, $\mathbf{u} \mapsto \mathbf{u}/U$, $\mathbf{v} \mapsto \mathbf{v}/U$, and $t \mapsto t/T$, and rearranging various terms, we write (2.1) as a system of first-order integro-differential equations

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A}_{\mathbf{u}}(\mathbf{y}, t), \\ &\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \kappa\mu^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}\frac{\mathbf{w}(s)}{\sqrt{t-s}}\,\mathrm{d}s\right) + \mu\mathbf{w} = -\mathbf{M}_{\mathbf{u}}(\mathbf{y}, t)\mathbf{w} + \mathbf{B}_{\mathbf{u}}(\mathbf{y}, t), \\ &\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0, \end{aligned} \tag{2.2}$$

with

$$\mathbf{w}(t) = \mathbf{v}(t) - \mathbf{u}(\mathbf{y}(t), t) - \frac{\gamma}{6}\mu^{-1}\Delta\mathbf{u}(\mathbf{y}(t), t), \tag{2.3a}$$

$$\begin{aligned} \mathbf{A}_{\mathbf{u}} &= \mathbf{u} + \frac{\gamma}{6}\mu^{-1}\Delta\mathbf{u}, \\ \mathbf{B}_{\mathbf{u}} &= \left(\frac{3R}{2} - 1\right)\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathbf{g}\right) + \left(\frac{R}{20} - \frac{1}{6}\right)\gamma\mu^{-1}\frac{\mathrm{D}}{\mathrm{D}t}\Delta\mathbf{u} \\ &\quad - \frac{\gamma}{6}\mu^{-1}\left[\nabla\mathbf{u} + \frac{\gamma}{6}\mu^{-1}\nabla\Delta\mathbf{u}\right]\Delta\mathbf{u}, \\ \mathbf{M}_{\mathbf{u}} &= \nabla\mathbf{u} + \frac{\gamma}{6}\mu^{-1}\nabla\Delta\mathbf{u}. \end{aligned} \tag{2.3b}$$

In deriving (2.2), we used the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\frac{\mathbf{w}(s)}{\sqrt{t-s}}\,\mathrm{d}s = \int_{t_0}^{t}\frac{\dot{\mathbf{w}}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{\mathbf{w}(t_0)}{\sqrt{t-t_0}},$$

obtained by differentiating and then integrating by parts (see, e.g., [29]).

The dimensionless parameters in (2.3) are defined as

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \qquad \mu = \frac{R}{\mathrm{St}}, \qquad \kappa = \sqrt{\frac{9R}{2}}, \qquad \gamma = \frac{9R}{2\mathrm{Re}}, \tag{2.4}$$

where the Stokes (St) and the Reynolds (Re) numbers are defined as

$$\mathrm{St} = \frac{2}{9}\left(\frac{a}{L}\right)^2 \mathrm{Re}, \qquad \mathrm{Re} = \frac{UL}{\nu}. \tag{2.5}$$

Note that the vector fields $\mathbf{A_u}, \mathbf{B_u} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^n$ and the tensor field $\mathbf{M_u} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ are known functions of the fluid velocity field $\mathbf{u}$.

Equation (2.3a) defines a simple one-to-one correspondence between the particle velocity $\mathbf{v}$ and the variable $\mathbf{w}$. Once a solution $(\mathbf{y}, \mathbf{w})$ of (2.2) is known, the particle velocity can be obtained as $\mathbf{v}(t) = \mathbf{w}(t) + \mathbf{u}(\mathbf{y}(t), t) + (\gamma\mu^{-1}/6)\Delta\mathbf{u}(\mathbf{y}(t), t)$. In the absence of the Faxén correction term $(\gamma\mu^{-1}/6)\Delta\mathbf{u}$, the variable $\mathbf{w} = \mathbf{v} - \mathbf{u}$ is the relative velocity between the particle and the fluid.

The integral term in (2.2) is proportional to the Riemann–Liouville fractional derivative of order $1/2$, which is defined as

$$\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\mathbf{w}(s)}{\sqrt{t-s}}\, \mathrm{d}s\right), \tag{2.6}$$

where $t \geq t_0$ [29]. Using this notation, we write the initial value problem (2.2) in the more compact form

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A_u}(\mathbf{y}, t), \\ &\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \kappa\mu^{1/2}\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} + \mu\mathbf{w} = -\mathbf{M_u}(\mathbf{y}, t)\mathbf{w} + \mathbf{B_u}(\mathbf{y}, t), \\ &\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0. \end{aligned} \tag{2.7}$$

2.2.2 Set-up and assumptions

We use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^m$ with $m \in \{n, 2n\}$. The induced operator norm of a square matrix acting on $\mathbb{R}^m$ is denoted by

$\|\cdot\|$. We denote the supremum norm of functions by $\|\cdot\|_\infty$.

For future use, we also define the function space

$$X_K^{t,h} = \{\mathbf{f} \in C\left([t, t+h]\,; \mathbb{R}^m\right) : \|\mathbf{f}\|_\infty \leq K\}. \tag{2.8}$$

Since $X_K^{t,h}$ is a closed subset of $C([t,t+h]; \mathbb{R}^m)$, the metric space $(X_K^{t,h}, \|\cdot\|_\infty)$ is a complete metric space.

For the Maxey–Riley equation (2.2), or its original form (2.1), to make sense, the partial derivatives of the fluid velocity $\partial_x^\alpha \mathbf{u}(\mathbf{x}, t)$ and $\partial_t \partial_x^\beta \mathbf{u}(\mathbf{x}, t)$, with $|\alpha| \leq 3$ and $|\beta| \leq 2$, must exist.

The Faxén corrections (the terms involving $\Delta \mathbf{u}$) are routinely neglected in practice [16, 17]. On neglecting the Faxén terms, the regularity assumption for the fluid velocity relaxes to the existence of the first order partial derivative with respect to space and time, that is, $|\alpha| \leq 1$ and $\beta = 0$. We do not neglect the Faxén terms in the work presented here.

For proving the global existence and uniqueness of mild solutions to the MR equation, we need the above partial derivatives to be uniformly bounded and Lipschitz continuous in space and time. In particular, we assume the following.

(H1) The velocity field $\mathbf{u}(\mathbf{x}, t)$ is smooth enough such that the partial derivatives $\partial_x^\alpha \mathbf{u}$ with $|\alpha| \leq 3$ and the mixed partial derivatives $\partial_t \partial_x^\beta \mathbf{u}$ with $|\beta| \leq 2$ defined over the domain $\mathcal{D} \times \mathbb{R}$ are uniformly bounded.

(H2) The velocity field $\mathbf{u}(\mathbf{x}, t)$ is smooth enough such that the partial derivatives $\partial_x^\alpha \mathbf{u}$ with $|\alpha| \leq 3$ and the mixed partial derivatives $\partial_t \partial_x^\beta \mathbf{u}$ with $|\beta| \leq 2$ defined over the domain $\mathcal{D} \times \mathbb{R}$ are uniformly Lipschitz continuous.

Remark 2.1 *Neglecting the Faxén terms, assumptions (H1) and (H2) relax, respectively, to the uniform boundedness and uniform Lipschitz continuity of the fluid velocity* $\mathbf{u}$ *and acceleration* $\frac{D\mathbf{u}}{Dt}$.

Assumption (H1) implies the existence of constants $L_A, L_B, L_M > 0$ such that

$$\|\mathbf{A_u}\|_\infty \leq L_A, \quad \|\mathbf{B_u}\|_\infty \leq L_B, \quad \|\mathbf{M_u}\|_\infty \leq L_M. \tag{2.9}$$

Assumption (H2) implies the existence of a constant $L_c > 0$ such that

$$\begin{aligned} |\mathbf{A_u}(\mathbf{y}_1, t) - \mathbf{A_u}(\mathbf{y}_2, t)| &\leq L_c|\mathbf{y}_1 - \mathbf{y}_2|, \\ |\mathbf{B_u}(\mathbf{y}_1, t) - \mathbf{B_u}(\mathbf{y}_2, t)| &\leq L_c|\mathbf{y}_1 - \mathbf{y}_2|, \\ \|\mathbf{M_u}(\mathbf{y}_1, t) - \mathbf{M_u}(\mathbf{y}_2, t)\| &\leq L_c|\mathbf{y}_1 - \mathbf{y}_2|, \end{aligned} \tag{2.10}$$

for all $\mathbf{y}_1$, $\mathbf{y}_2 \in \mathcal{D}$ and all $t \in \mathbb{R}$. The supremum norms in (2.9) are taken over all $(\mathbf{y}, t) \in \mathcal{D} \times \mathbb{R}$.

Farazmand and Haller [3] proved the following local existence and uniqueness result.

Theorem 2.1 (Farazmand & Haller, [3]) *Assume that (H1) and (H2) hold. For any $(\mathbf{y}_0, \mathbf{w}_0) \in \mathcal{D} \times \mathbb{R}^n$, there exists a time increment $\delta t > 0$ such that, over the time interval $[t_0, t_0 + \delta t)$, the Maxey–Riley equation (2.7) has a unique solution $(\mathbf{y}(t), \mathbf{w}(t))$ satisfying $(\mathbf{y}(t_0), \mathbf{w}(t_0)) = (\mathbf{y}_0, \mathbf{w}_0)$.*

2.2.3 The Maxey–Riley equation does not generate a dynamical system

For ordinary differential equations, one may construct global solutions by continuation. In particular, given a local solution $(\mathbf{y}(t), \mathbf{w}(t))$ existing on a time interval $[t_0, t_0 + \Delta_1)$, one shows that the solution does not blow up at $t = t_0 + \Delta_1$. Then initializing the ordinary differential equation from time $t = t_0 + \Delta_1$ with initial condition $(\mathbf{y}(t_0 + \Delta_1), \mathbf{w}(t_0 + \Delta_1))$, the local existence and uniqueness result is reapplied to show that the solution can be extended to an interval $[t_0, t_0 + \Delta_1 + \Delta_2)$. Repeating the above steps, the solution can be extended to a time interval $[t_0, t_0 + \Delta_1 + \Delta_2 + \Delta_3 + \cdots)$. Finally, one shows that the infinite series $\Delta_1 + \Delta_2 + \Delta_3 + \cdots$ diverges and infers global existence and uniqueness.

This continuation argument assumes that the flow map $\mathbf{F}_{t_0}^t : (\mathbf{y}_0, \mathbf{w}_0) \mapsto (\mathbf{y}(t), \mathbf{w}(t))$ has the semi-group property $\mathbf{F}_{t_0}^t = \mathbf{F}_{t_1}^t \circ \mathbf{F}_{t_0}^{t_1}$ for all $t_0 < t_1 < t$. Due to the fractional derivative, however, the flow map of the Maxey–Riley equation (2.7) is not a semi-group.

To see this, consider the solution $(\mathbf{y}(t), \mathbf{w}(t))$ starting from $(\mathbf{y}_0, \mathbf{w}_0)$ at time t_0. Due to the Basset history force, that is, the fractional derivative in (2.7), the trajectory $(\mathbf{y}(t), \mathbf{w}(t))$ for $t > t_1$ is influenced by its entire past history. A trajectory initialized from $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ is, however, ignorant of this history and therefore will follow a different path (see figure 2.1 for an illustration).

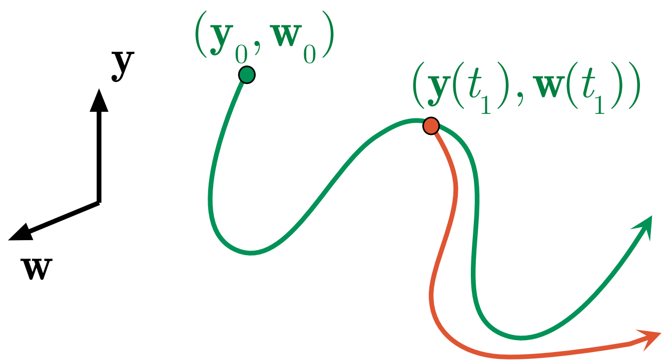

Figure 2.1 – A trajectory $(\mathbf{y}(t), \mathbf{w}(t))$ of the MR equation (2.7) initialized from $(\mathbf{y}_0, \mathbf{w}_0)$ and passing through $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ at time t_1 (green curve). A trajectory initialized from $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ at time t_1 (red curve) does not follow the trajectory $(\mathbf{y}(t), \mathbf{w}(t))$.

As a result, the usual continuation methods for ordinary differential equations do not apply here. In section 2.4.1, we construct a specific continuation suitable for the MR equation.

2.2.4 Rescaling time

We introduce a rescaling of time that further simplifies the forthcoming analysis. Dividing the $\mathbf{w}$ component of equation (2.2) by μ and letting

$\epsilon := \frac{1}{\mu}$, we get

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A}_{\mathbf{u}}(\mathbf{y}, t), \\
&\epsilon\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \epsilon^{1/2}\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} + \mathbf{w} = -\epsilon\mathbf{M}_{\mathbf{u}}(\mathbf{y}, t)\mathbf{w} + \epsilon\mathbf{B}_{\mathbf{u}}(\mathbf{y}, t), \\
&\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0.
\end{aligned} \tag{2.11}$$

Note that by (2.5), $\epsilon = \frac{St}{R} = \frac{2}{9R}\left(\frac{a}{L}\right)^2 \mathrm{Re}$. Since the Maxey–Riley equation holds for small particles ($a \ll L$), ϵ is necessarily a small and positive parameter: $0 < \epsilon \ll 1$. Thus the limit $\epsilon \to 0$ ($a \to 0$) describes the limit of infinitesimally small particles.

Rescaling time as $t = t_0 + \epsilon\tau$, we have

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{\mathbf{y}}}{\mathrm{d}\tau} = \epsilon\left[\tilde{\mathbf{w}} + \tilde{\mathbf{A}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau)\right], \\
&\frac{\mathrm{d}\tilde{\mathbf{w}}}{\mathrm{d}\tau} + \kappa\frac{\mathrm{d}^{1/2}\tilde{\mathbf{w}}}{\mathrm{d}\tau^{1/2}} + \tilde{\mathbf{w}} = \epsilon\left[-\tilde{\mathbf{M}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau)\tilde{\mathbf{w}} + \tilde{\mathbf{B}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau)\right], \\
&\tilde{\mathbf{y}}(0) = \mathbf{y}_0, \quad \tilde{\mathbf{w}}(0) = \mathbf{w}_0,
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\tilde{\mathbf{y}}(\tau) &= \mathbf{y}(t_0 + \epsilon\tau), \\
\tilde{\mathbf{w}}(\tau) &= \mathbf{w}(t_0 + \epsilon\tau), \\
\tilde{\mathbf{A}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau) &= \mathbf{A}_{\mathbf{u}}(\mathbf{y}, t_0 + \epsilon\tau), \\
\tilde{\mathbf{B}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau) &= \mathbf{B}_{\mathbf{u}}(\mathbf{y}, t_0 + \epsilon\tau), \\
\tilde{\mathbf{M}}_{\mathbf{u}}(\tilde{\mathbf{y}}, \tau) &= \mathbf{M}_{\mathbf{u}}(\mathbf{y}, t_0 + \epsilon\tau),
\end{aligned}$$

and

$$\frac{\mathrm{d}^{1/2}\tilde{\mathbf{w}}}{\mathrm{d}\tau^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\tilde{\mathbf{w}}(s)}{\sqrt{\tau - s}}\,\mathrm{d}s\right).$$

The above rescaling of time has been previously used [24–26] for the asymptotic analysis of the MR equation without memory. It allows us to treat equation (2.12) as a *regular* perturbation problem with respect to ϵ, as opposed to treating equation (2.11) as a *singular* perturbation problem with respect to ϵ. To see the singular nature of the perturbation, divide the $\mathbf{w}(t)$ equation of (2.11) by ϵ and take the limit $\epsilon \to 0$: the limit will be unbounded. The regularized $\epsilon \to 0$ limit of (12) is unphysical, corresponding to an inertial particle of zero radius. However, equation (2.12) is physically meaningful for any $\epsilon > 0$.

Note that a unique solution of the initial value problem (IVP) (2.12) over the time interval $[0, \delta)$ exists if and only if the unscaled IVP (2.7) has a unique solution over the time interval $[t_0, t_0 + \epsilon\delta)$. Therefore, in the following, we study the IVP (2.12). We will first analyze the solution of the IVP (2.12) in the limit $\epsilon = 0$, and then use this solution to study the IVP (2.12) for $\epsilon > 0$. For notational simplicity, we omit the tilde signs from all the variables.

2.3 Asymptotic behavior

2.3.1 $\epsilon = 0$ limit

We start with the limit $\epsilon = 0$ of equation (2.12), which as discussed in section 2.2.4 is unphysical. In this limit, $\mathbf{y}(\tau) = \mathbf{y}_0$ remains constant for all times, and $\mathbf{w}(\tau)$ becomes

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} + \kappa \frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} + \mathbf{w} = \mathbf{0}, \\ &\mathbf{w}(0) = \mathbf{w}_0, \end{aligned} \tag{2.13}$$

where κ is the dimensionless parameter defined by (2.4). Equation (2.13) is a linear equation tractable by Laplace transforms [29, 34]. This leads to the following result.

Theorem 2.2 *The general solution of* (2.13) *is given by* $\mathbf{w}(\tau; \mathbf{w}_0) = \psi_\kappa(\tau)\mathbf{w}_0$, *where the positive, scalar function* $\psi_\kappa : [0, \infty) \to \mathbb{R}^+$ *has the following properties.*

1. *ψ_κ is given by the inverse Laplace transform*

$$\psi_\kappa(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)}\right](\tau), \tag{2.14}$$

where

$$\lambda_\pm = \frac{\kappa \pm \sqrt{\kappa^2-4}}{2}.$$

2. *ψ_κ obeys the asymptotic decay rate*

$$\psi_\kappa(\tau) \sim \frac{\kappa}{2\sqrt{\pi}}\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right) \quad as \quad \tau \to \infty. \tag{2.15}$$

3. *There is a differentiable function $\phi_\kappa : [0,\infty) \to \mathbb{R}^+$ such that $\psi_\kappa = -\phi_\kappa'$.*

4. *The functions ψ_κ and ϕ_κ are smooth over $\in (0,\infty)$ and completely monotonic decreasing, i.e.,*

$$(-1)^j\psi_\kappa^{(j)}(\tau) \geq 0, \quad (-1)^j\phi_\kappa^{(j)}(\tau) \geq 0, \quad j = 0,1,2,\cdots, \quad \forall\tau > 0.$$

5. *$\psi_\kappa(0) = 1$ and $\phi_\kappa(0) = 1$.*

Proof See appendix 2.A for the proof of 1, 2, and the calculation of ψ_κ. For the proof of 3, 4, and 5, see the properties demonstrated for u_δ and u_0 (here, ψ_κ and ϕ_κ, respectively) in Gorenflo and Mainardi [34]. □

Figure 2.2 shows the functions ϕ_κ and ψ_κ computed by numerically inverting their Laplace transforms. It follows from properties 2 and 3 of theorem 2.2 that ϕ_κ decays asymptotically as $\tau^{-1/2}$, as confirmed by the numerics.

Since the properties of theorem 2.2 hold for any $\kappa > 0$, we omit the dependence of ψ_κ and ϕ_κ on κ and write ψ and ϕ, respectively.

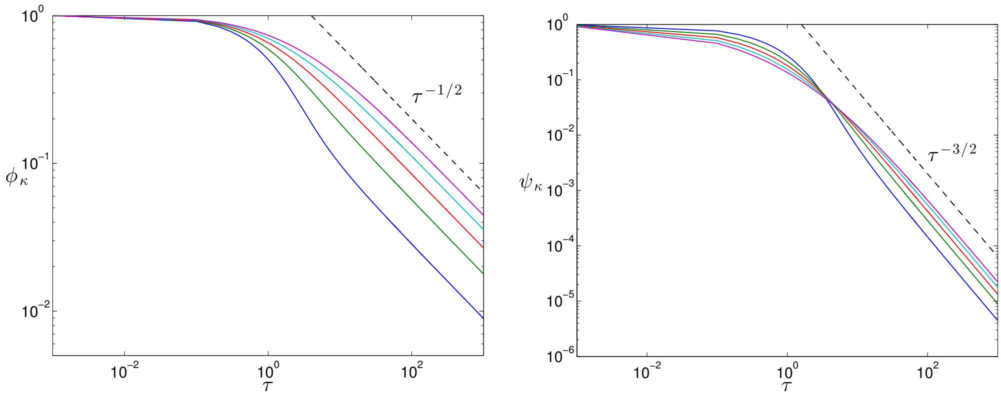

Figure 2.2 – The functions ϕ_κ and $\psi_\kappa = -\phi'_\kappa$. The functions are evaluated for $\kappa = 0.5$ (blue), $\kappa = 1$ (green), $\kappa = 1.5$ (red), $\kappa = 2$ (cyan) and $\kappa = 2.5$ (magenta).

2.3.2 $\epsilon > 0$ case

Now we analyze the general case of $\epsilon > 0$, i.e.,

$$\begin{aligned} &\frac{d\mathbf{y}}{d\tau} = \epsilon\left[\mathbf{w} + \mathbf{A_u}(\mathbf{y},\tau)\right], \\ &\frac{d\mathbf{w}}{d\tau} + \kappa\frac{d^{1/2}\mathbf{w}}{d\tau^{1/2}} + \mathbf{w} = \epsilon\left[-\mathbf{M_u}(\mathbf{y},\tau)\mathbf{w} + \mathbf{B_u}(\mathbf{y},\tau)\right], \\ &\mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{w}(0) = \mathbf{w}_0, \end{aligned} \tag{2.16}$$

which is equation (2.12) with tilde signs omitted. Solutions of (2.16) satisfy the integral equations

$$\begin{aligned} \mathbf{y}(\tau) &= \mathbf{y}_0 + \epsilon\int_0^\tau \mathbf{w}(s) + \mathbf{A_u}(\mathbf{y}(s),s)\, ds, \\ \mathbf{w}(\tau) &= \psi(\tau)\mathbf{w}_0 + \epsilon\int_0^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\mathbf{y}(s),s)\mathbf{w}(s) + \mathbf{B_u}(\mathbf{y}(s),s)\right] ds, \end{aligned} \tag{2.17}$$

where $\psi(\tau)$ satisfies the properties listed in theorem 2.2.

This integral equation is essentially a variation of constants formula. The $\mathbf{y}$ equation in (2.17) is obtained by formal integration of the $d\mathbf{y}/d\tau$ equation of (2.16). For the $\mathbf{w}$ equation, let $\mathbf{W}(s)$ denote the Laplace transform of $\mathbf{w}(\tau)$. Taking the Laplace transform of (2.16) yields

$$\mathbf{W}(s) = \frac{\mathbf{w}_0 + \mathcal{L}\left[-\mathbf{M}_\mathbf{u}(\mathbf{y}(\tau),\tau)\mathbf{w}(\tau) + \mathbf{B}_\mathbf{u}(\mathbf{y}(\tau),\tau)\right](s)}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)}.$$

Taking the inverse Laplace transform, we get the $\mathbf{w}$-component of equation (2.17), where $\psi(\tau)$ is given by (2.14).

Definition 2.1 *A mild solution of the IVP (2.16) is a function $(\mathbf{y},\mathbf{w}) : [0,\delta) \to \mathbb{R}^{2n}$ that solves the integral equation (2.17). The existence time $\delta > 0$ may be infinite.*

Using the integral equation (2.17), we find an upper bound for $|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|$ and its asymptotic limit.

Theorem 2.3 *Assume that (H1) holds and $\epsilon < 1/L_M$. Let $(\mathbf{y},\mathbf{w}) : [0,\delta) \to \mathbb{R}^{2n}$ be a mild solution of (2.16) where $[0,\delta)$ is the maximal interval of existence of such solutions.*

(i) An explicit envelope for $|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|$ is given by

$$\begin{aligned} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq |\mathbf{w}_0| &\left[\sum_{j=1}^{\infty}(\epsilon L_M)^{j-1}\psi^{*j}(\tau)\right] \\ &+ \epsilon L_B\left(1-\phi(\tau)\right) + \frac{\epsilon^2 L_M L_B}{1-\epsilon L_M}, \end{aligned} \tag{2.18}$$

where ψ^{*j} is the j-fold convolution of ψ. Moreover, the series converges uniformly and is bounded for all τ.

(ii) $|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|$ is bounded for all $\tau \in [0,\delta)$. Specifically,

$$\sup_{0\leq\tau<\delta} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{|\mathbf{w}_0| + \epsilon L_B}{1-\epsilon L_M}. \tag{2.19}$$

(iii) If $\delta = \infty$, the asymptotic limit of $\mathbf{w}$ satisfies

$$\limsup_{\tau\to\infty} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{\epsilon L_B}{1-\epsilon L_M}. \tag{2.20}$$

Proof See appendix 2.B. □

In deriving the upper envelope (2.18) and the subsequent upper bounds (2.19) and (2.20), we have made several upper estimates. The natural question arising is how sharp these estimates are. In the following section, among other things, we show with a numerical example that these bounds are sharp by showing that they can be saturated.

2.3.3 Numerical verification

We illustrate the results of theorem 2.3 with an example. For the fluid flow, we use the double gyre model of Shadden et al. [33]. It is a two-dimensional velocity field with the stream function

$$\mathcal{H}(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y), \tag{2.21}$$

where

$$f(x, t) = \alpha \sin(\omega t) x^2 + (1 - 2\alpha \sin(\omega t)) x.$$

We let $A = 0.1$, $\omega = \pi$ and $\alpha = 0.01$.

The Hamiltonian $\mathcal{H}$ defines the velocity field $\mathbf{u} = (-\partial_y \mathcal{H}, \partial_x \mathcal{H})^\top$ which we use to solve the initial value problem (2.7) using the numerical scheme developed by Daitche [35]. We will neglect the Faxén corrections, such that $\mathbf{A_u} = \mathbf{u}$, $\mathbf{B_u} = \left(\frac{3R}{2} - 1\right) \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ and $\mathbf{M_u} = \nabla \mathbf{u}$ (recall, however, that the main results also hold with the Faxén corrections).

For the parameters of the inertial particle, we let $\mathrm{St} = R/100$ resulting in $\mu = 100$ (or $\epsilon = 0.01$). This corresponds to a small inertial particle (with respect to the underlying flow) since by equation (2.5) the Stokes number is proportional to the square of the particle's radius. Three values of R are considered here: $R = 2/3$ (neutrally buoyant particle, $\rho_f = \rho_p$), $R = 1/3$ (aerosol, $\rho_f < \rho_p$) and $R = 1$ (bubble, $\rho_f > \rho_p$). In each case, we release 15 trajectories with initial conditions $\mathbf{y}_0$ uniformly distributed in the domain $[0.2, 1.8] \times [0.2, 0.8]$ (i.e., $\mathbf{y}_0 \in \{0.2, 0.6, 1.0, 1.4, 1.8\} \times \{0.2, 0.5, 0.8\}$) and identical initial relative

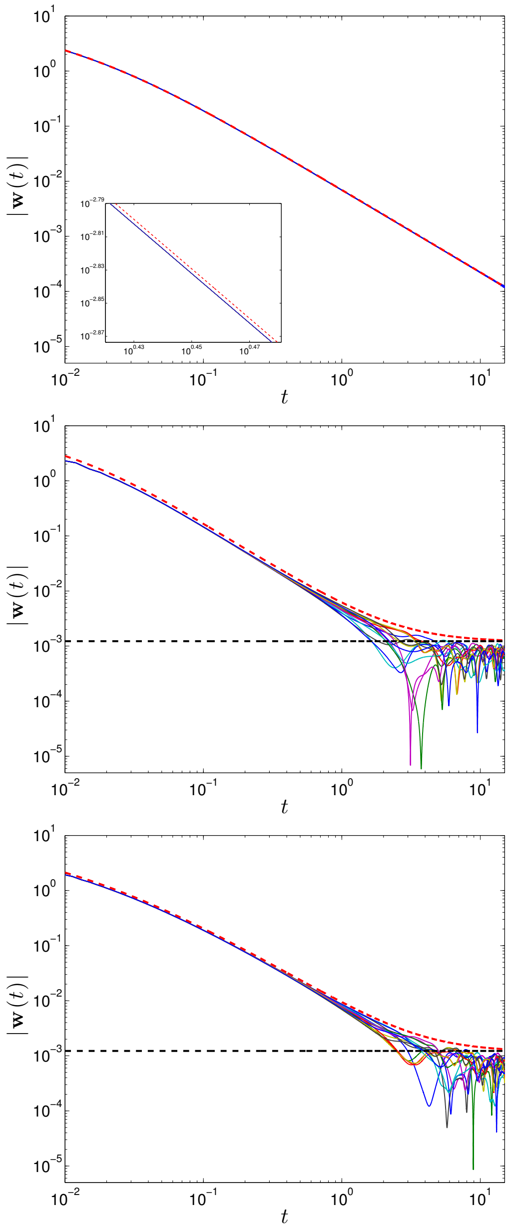

Figure 2.3 – The decay of the relative velocity magnitude $|\mathbf{w}(t)|$ for $R = 2/3$ (a), $R = 1/3$ (b) and $R = 1$ (c). The dashed red lines mark the analytic envelope from part (i) of theorem 2.3. The dashed black lines mark the asymptotic upper bound of $|\mathbf{w}|$, i.e., $\epsilon L_B/(1 - \epsilon L_M)$. The initial value of $|\mathbf{w}(t)|$ is $10\sqrt{2}$ in all cases. In order to focus on the asymptotics, we only plot the graphs for $t \geq 10^{-2}$.

velocities $\mathbf{w}_0 = (10, 10)^\top$. We picked large initial velocities in this example to show the algebraic decay of $|\mathbf{w}|$ more clearly.

We take the most conservative choices of the upper bounds L_B and L_M, i.e., $L_B = \|\mathbf{B}_\mathbf{u}\|_\infty$ and $L_M = \|\mathbf{M}_\mathbf{u}\|_\infty$. For the neutrally buoyant particle, i.e., $R = 2/3$, $\mathbf{B}_\mathbf{u}$ vanishes identically, resulting in $L_B = 0$. The norm $\|M_\mathbf{u}\|_\infty$ is, however, independent of R and we have $L_M \simeq 1.4237$. Theorem 2.3 therefore implies that for a neutrally buoyant particle, $|\mathbf{w}(t)|$ must decay to zero asymptotically, which agrees with the numerical result (see figure 2.3a). Physically, this implies that the inertial particle trajectory converges to a fluid trajectory. In the case of neutrally buoyant particles, the theoretical envelope and the numerical solutions almost coincide. This is because for $R = 2/3$ the two terms proportional to L_B vanish in the estimate (2.18), apparently making the upper bound close to optimal. A close-up view is shown in the inset of figure 2.3a.

Interestingly, for the neutrally buoyant particle the evolution of the relative velocity magnitude $|\mathbf{w}|$ seems to be independent of the initial positions $\mathbf{y}_0$ as all 15 curves coincide in figure 2.3a.

For the bubble ($R = 1$) and the aerosol ($R = 1/3$), we have $L_B \simeq 0.1207$ and $L_M \simeq 1.4237$. The resulting envelope (2.18) and the asymptotic upper bound $\epsilon L_B/(1 - \epsilon L_M)$ are also shown (red and black dashed curves, respectively) which shows a perfect agreement with the numerical results. In plotting the envelopes, $\mathcal{O}(\epsilon^2)$-terms are neglected. The numerical solutions come very close to the analytic envelope of part (i) of theorem 2.3, indicating the tightness of the estimates.

The upper envelope (2.18) depends on the functions ϕ and ψ, which in turn depend on the parameter $\kappa = \sqrt{9R/2}$. The parameter R is governed by the ratio of the particle density ρ_p over the fluid density ρ_f. As this ratio varies the upper envelope also changes. Owing to the algebraic transient decay of ϕ and ψ (see figure 2.2), however, the envelope exhibits an algebraic decay regardless of the value of R. Figure 2.4 shows the behavior of the upper envelope (neglecting $\mathcal{O}(\epsilon^2)$-terms) for

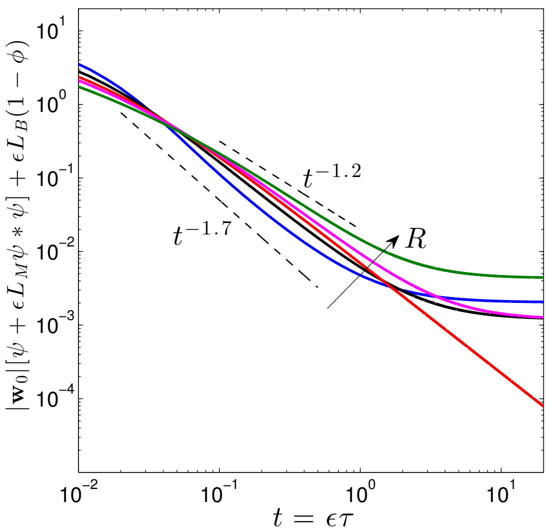

Figure 2.4 – The upper envelope (2.18), neglecting $\mathcal{O}(\epsilon^2)$-terms, for $R = 1/10$ (blue), $R = 1/3$ (black), $R = 2/3$ (red), $R = 1$ (magenta) and $R = 19/10$ (green).

the double gyre parameters and various values of R. For neutrally buoyant particle ($R = 2/3$) there is a monotonic decay with the algebraic rate $t^{-3/2}$. For other values of R the envelope decays to the asymptotic upper bound. There is still a transient algebraic decay whose rate varies, depending on the parameter R, between $t^{-1.7}$ and $t^{-1.2}$.

2.4 Global existence and uniqueness

In this section, we prove the global existence and uniqueness of mild solutions to the full Maxey–Riley equation (2.1) with the Faxén correction terms. In particular, we show that the equivalent reformulation (2.16) admits unique mild solutions for all times, that is, the integral equations (2.17) have a unique solution over $\mathbb{R}^+$.

The existence of a unique local solution follows from theorem 2.1. Specifically, the integral equation (2.17) has a unique solution over the time interval $[0, \delta t/\epsilon)$, where δt is the same time window as in theorem 2.1, with the ϵ appearing due to the rescaling $t = t_0 + \epsilon\tau$ as introduced in section 2.2.4. For notational convenience, we let $\delta = \delta t/\epsilon$.

As discussed in section 2.2.3, the usual continuation methods used for ordinary differential equations do not apply to fractional differential equations. Therefore, we construct a specific continuation method suitable for the MR equation, which is based on the continuation method presented in the work of Kou et al. [36] for a different class of fractional differential equations. We then show that this continuation can be repeated indefinitely to extend the solutions to the time interval $[0, \infty)$. Our approach can be summarized in the following steps.

Step 1. Show that the local solution of the integral equation (2.17), defined on $[0, \delta)$, is well defined at time $\tau = \delta$.

Step 2. Define a suitable integral operator $\mathbf{F}$ over an appropriate complete metric space whose fixed points extend the local solution of (2.17) from $[0, \delta)$ to $[0, \delta + h)$, for a suitable constant $h > 0$.

Step 3. Show that the operator $\mathbf{F}$ has at least one fixed point.

Step 4. Show that this continuation is unique.

Step 5. Show that one can repeat steps 1 to 4 indefinitely with the same continuation window h. That is the local solution of (2.17) can be continued uniquely to $\mathbb{R}^+$.

The above steps prove the following global existence and uniqueness theorem.

Theorem 2.4 *Assume that (H1) and (H2) hold and $\epsilon < 1/L_M$. Then the Maxey–Riley equation has unique, continuous, mild solutions. That is, for any $(\mathbf{y}_0, \mathbf{w}_0) \in \mathbb{R}^{2n}$, there exists a unique, continuous function $(\mathbf{y}, \mathbf{w}) : [0, \infty) \to \mathbb{R}^{2n}$ satisfying (2.17) and $(\mathbf{y}(0), \mathbf{w}(0)) = (\mathbf{y}_0, \mathbf{w}_0)$.*

2.4.1 Continuation of the local solution

We denote the local solution of the Maxey–Riley equation, whose existence and uniqueness is guaranteed by theorem 2.1, by $\mathbf{z}_{loc} = (\mathbf{y}_{loc}, \mathbf{w}_{loc})$.

We first show that this local solution defined on $[0, \delta)$ is well defined at $\tau = \delta$.

Lemma 2.1 *The local solution $\mathbf{z}_{loc} : [0, \delta) \to \mathbb{R}^{2n}$ to the Maxey–Riley equation is well-defined at $\tau = \delta$ and the limit $\lim_{\tau \to \delta^-} \mathbf{z}_{loc}(\tau)$ is given by*

$$\mathbf{z}_{loc}(\delta) = \begin{pmatrix} \mathbf{y}_0 + \epsilon \int_0^\delta \mathbf{w}_{loc}(s) + \mathbf{A}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\, \mathrm{d}s \\ \psi(\delta)\mathbf{w}_0 + \epsilon \int_0^\delta \psi(\delta - s) \left[-\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\mathbf{w}_{loc}(s) + \mathbf{B}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s) \right] \mathrm{d}s \end{pmatrix}. \tag{2.22}$$

Proof See appendix 2.C. □

Let $(\mathbf{y}_{loc}, \mathbf{w}_{loc}) : [0, \delta) \to \mathbb{R}^{2n}$ be the local solution of (2.17) whose existence and uniqueness is guaranteed by theorem 2.1. Define

$$\mathbf{y}(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}(\tau), \tag{2.23a}$$

$$\mathbf{w}(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}(\tau), \tag{2.23b}$$

where $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$ is the indicator function of the set $A \subset \mathbb{R}$. Note that for $\tau \in [0, \delta)$, $(\mathbf{y}, \mathbf{w})$ coincides with the local solution $(\mathbf{y}_{loc}, \mathbf{w}_{loc})$. Assume that $(\mathbf{y}, \mathbf{w})$ is a continuation of this local solution to $[0, \delta + h)$. On substitution in (2.17), we have

$$\begin{aligned} \boldsymbol{\eta}(\tau) &= \mathbf{y}_0 + \epsilon \int_0^\delta \mathbf{w}_{loc}(s) + \mathbf{A}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\, \mathrm{d}s + \epsilon \int_\delta^\tau \boldsymbol{\eta}(s) + \mathbf{A}_{\mathbf{u}}(\boldsymbol{\xi}(s), s)\, \mathrm{d}s, \\ \boldsymbol{\xi}(\tau) &= \psi(\tau)\mathbf{w}_0 + \epsilon \int_0^\delta \psi(\tau - s) \left[-\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\mathbf{w}_{loc}(s) + \mathbf{B}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s) \right] \mathrm{d}s \\ &\quad + \epsilon \int_\delta^\tau \psi(\tau - s) \left[-\mathbf{M}_{\mathbf{u}}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s) + \mathbf{B}_{\mathbf{u}}(\boldsymbol{\xi}(s), s) \right] \mathrm{d}s, \end{aligned} \tag{2.24}$$

for $\tau \in [\delta, \delta + h)$.

Therefore, $(\mathbf{y}, \mathbf{w})$ solves the integral equation (2.17) and hence is a mild solution to the MR equation if and only if the integral equation (2.24) has a solution. To show that such a solution exists, we solve the following fixed point problem. Let $\mathbf{\Phi} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in X_K^{\delta,h}$. Define the operator $\mathbf{F} : X_K^{\delta,h} \to C([\delta, \delta + h); \mathbb{R}^{2n})$ by

$$\begin{aligned} &(\mathbf{F\Phi})(\tau) \\ &\quad = \mathbf{\Phi}_0(\tau) + \begin{pmatrix} \epsilon \int_\delta^\tau \boldsymbol{\eta}(s) + \mathbf{A_u}(\boldsymbol{\xi}(s), s)\ \mathrm{d}s \\ \epsilon \int_\delta^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s) + \mathbf{B_u}(\boldsymbol{\xi}(s), s)\right]\ \mathrm{d}s \end{pmatrix}, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} &\mathbf{\Phi}_0(\tau) \\ &\quad = \begin{pmatrix} \mathbf{y}_0 + \epsilon \int_0^\delta \mathbf{w}_{loc}(s) + \mathbf{A_u}(\mathbf{y}_{loc}(s), s)\ \mathrm{d}s \\ \psi(\tau)\mathbf{w}_0 + \epsilon \int_0^\delta \psi(\tau - s)\left[-\mathbf{M_u}(\mathbf{y}_{loc}(s), s)\mathbf{w}_{loc}(s) + \mathbf{B_u}(\mathbf{y}_{loc}(s), s)\right]\ \mathrm{d}s \end{pmatrix}. \end{aligned} \tag{2.26}$$

Note that $\mathbf{\Phi}_0$ depends only on the local solution $(\mathbf{y}_{loc}, \mathbf{w}_{loc})$ of the Maxey–Riley equation and hence is independent of $\mathbf{\Phi}$. We show that the operator $\mathbf{F}$ maps $X_K^{\delta,h}$ to itself (with K and h to be determined) and has a unique fixed point.

2.4.2 Existence of the continuation

Proposition 2.1 *Assume that (H1) holds. There exist constants $h, K > 0$ such that the operator $\mathbf{F}$ defined in (2.25) maps $X_K^{\delta,h}$ to itself and has at least one fixed point.*

Proof For any $h, K > 0$ and $\mathbf{\Phi} \in X_K^{\delta,h}$ the function $\mathbf{F\Phi} : [\delta, \delta + h) \to \mathbb{R}^{2n}$ is clearly continuous, that is, $\mathbf{F\Phi} \in C\left([\delta, \delta + h); \mathbb{R}^{2n}\right)$. We choose $h, K > 0$ such that $\mathbf{F\Phi} \in X_K^{\delta,h}$, i.e., $\|\mathbf{F\Phi}\|_\infty \leq K$. To this end, note that for any

$h > 0$ and $\tau \in [\delta, \delta + h)$, we have

$$|(\mathbf{F\Phi})(\tau)| \leq |\mathbf{\Phi}_0(\tau)| + \epsilon \int_{\delta}^{\delta+h} |\boldsymbol{\eta}(s)| + |\mathbf{A}_{\mathbf{u}}(\boldsymbol{\xi}(s), s)| \,\mathrm{d}s \\ + \epsilon \int_{\delta}^{\delta+h} \psi(\tau - s) \left[|\mathbf{M}_{\mathbf{u}}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s)| + |\mathbf{B}_{\mathbf{u}}(\boldsymbol{\xi}(s), s)|\right] \,\mathrm{d}s.$$

Take the supremum over $\tau \in [\delta, \delta + h)$ and use the bounds on $\|\mathbf{M}_{\mathbf{u}}\|_\infty$, $\|\mathbf{B}_{\mathbf{u}}\|_\infty$, $\|\mathbf{A}_{\mathbf{u}}\|_\infty$, $|\mathbf{w}(\tau)|$, $\|\psi\|_\infty$, and $\|\boldsymbol{\eta}\|_\infty$ to get

$$\begin{aligned} \|\mathbf{F\Phi}\|_\infty &\leq \|\mathbf{\Phi}_0\|_\infty + \epsilon \int_{\delta}^{\delta+h} \|\boldsymbol{\eta}\|_\infty + \|\mathbf{A}_{\mathbf{u}}\|_\infty \,\mathrm{d}s \\ &\quad + \epsilon \int_{\delta}^{\delta+h} \left[\|\mathbf{M}_{\mathbf{u}}\|_\infty \|\boldsymbol{\eta}\|_\infty + \|\mathbf{B}_{\mathbf{u}}\|_\infty\right] \,\mathrm{d}s \\ &\leq \|\mathbf{\Phi}_0\|_\infty + \epsilon h \left(K + L_A\right) + \epsilon h \left(L_M K + L_B\right). \end{aligned}$$

For

$$h \leq \frac{1}{2\epsilon \left(L_M + 1\right)},$$

we have

$$\|\mathbf{F\Phi}\|_\infty \leq \|\mathbf{\Phi}_0\|_\infty + \frac{K}{2} + \frac{L_B + L_A}{2\left(L_M + 1\right)}.$$

Since $\mathbf{\Phi}_0 : [0, \infty) \to \mathbb{R}^{2n}$ is a continuous function, there exists a finite constant $K' > 0$ such that

$$\|\mathbf{\Phi}_0\|_\infty := \sup_{\delta \leq \tau < \delta + h} |\mathbf{\Phi}_0(\tau)| = K'$$

Choosing

$$K \geq \left[K' + \frac{L_B + L_A}{2\left(L_M + 1\right)}\right],$$

we have $\|\mathbf{F\Phi}\|_\infty \leq K$.

In short, with any $h, K > 0$ satisfying

$$h \leq \frac{1}{2\epsilon \left(L_M + 1\right)}, \quad K = K' + \frac{L_B + L_A}{2\left(L_M + 1\right)}, \tag{2.27}$$

the operator **F** maps $X_K^{\delta,h}$ to itself.

To prove the existence of a fixed point for the operator $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$, we use Schauder's fixed point theorem:

Theorem 2.5 (Schauder's Fixed Point theorem) *Let X be a real space, $D \subset X$ nonempty, closed, bounded, and convex. Let $\mathcal{F} : D \to D$ be a continuous, compact operator. Then $\mathcal{F}$ has a fixed point.*

The space $X_K^{\delta,h}$ is nonempty, closed, bounded, and convex. To apply Schauder's fixed point theorem, therefore, it remains to show that $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is continuous and compact. For this, we need the following lemma.

Lemma 2.2 *The operator **F** is continuous and maps $X_K^{\delta,h}$ to a family of equicontinuous functions in $X_K^{\delta,h}$.*

Proof The proof of the continuity of $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is straightforward and is therefore omitted here. We prove the equicontinuity of its range in appendix 2.D. □

By Arzelà-Ascoli theorem, therefore, the operator $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is compact. Hence, **F** satisfies all the conditions of Schauder's theorem and has at least one fixed point. This proves proposition 2.1. □

2.4.3 Uniqueness of the continuation

We now show that the continuation constructed in sections 2.4.1 and 2.4.2 is unique.

Proposition 2.2 *Assume that (H1) and (H2) hold and $\epsilon < 1/L_M$. There exists $h > 0$ such that the continuation (2.23) of the local solution to the MR equation is unique.*

Proof Suppose $(\mathbf{y}_1, \mathbf{w}_1)$ and $(\mathbf{y}_2, \mathbf{w}_2)$ are two different continuations of the local solution of (2.17) from $[0, \delta)$ to $[\delta, \delta + h)$. That is

$$\mathbf{y}_1(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}_1(\tau),$$

$$\mathbf{w}_1(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}_1(\tau),$$

and

$$\mathbf{y}_2(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}_2(\tau),$$

$$\mathbf{w}_2(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{loc}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}_2(\tau),$$

where, as discussed in section 2.4.1, $(\boldsymbol{\xi}_i, \boldsymbol{\eta}_i)$ solve the integral equations

$$\begin{pmatrix} \boldsymbol{\xi}_i(\tau) \\ \boldsymbol{\eta}_i(\tau) \end{pmatrix} = \boldsymbol{\Phi}_0(\tau) + \epsilon \begin{pmatrix} \int_\delta^\tau \boldsymbol{\eta}_i(s) + \mathbf{A_u}(\boldsymbol{\xi}_i(s), s)\ \mathrm{d}s \\ \int_\delta^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\boldsymbol{\xi}_i(s), s)\boldsymbol{\eta}_i(s) + \mathbf{B_u}(\boldsymbol{\xi}_i(s), s)\right]\ \mathrm{d}s \end{pmatrix}, \tag{2.30}$$

for $i \in \{1, 2\}$.

Define $\boldsymbol{\Phi}_i = (\boldsymbol{\xi}_i, \boldsymbol{\eta}_i)$ and bound $|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2|$ by

$$\begin{aligned} |\boldsymbol{\Phi}_1(\tau) - \boldsymbol{\Phi}_2(\tau)| \le \epsilon \int_\delta^{\delta+h} &|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)| + |\mathbf{A_u}(\boldsymbol{\xi}_1(s), s) - \mathbf{A_u}(\boldsymbol{\xi}_2(s), s)|\ \mathrm{d}s \\ &+ \epsilon \int_\delta^{\delta+h} |\psi(\tau - s)| \left(|\mathbf{M_u}(\boldsymbol{\xi}_1(s), s)(\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s))|\right. \\ &+ \left.|\boldsymbol{\eta}_2(s)||\mathbf{M_u}(\boldsymbol{\xi}_1(s), s) - \mathbf{M_u}(\boldsymbol{\xi}_2(s), s)|\right)\ \mathrm{d}s \\ &+ \epsilon \int_\delta^{\delta+h} |\psi(\tau - s)||\mathbf{B_u}(\boldsymbol{\xi}_1(s), s) - \mathbf{B_u}(\boldsymbol{\xi}_2(s), s)|\ \mathrm{d}s, \end{aligned} \tag{2.31}$$

where we wrote $|\mathbf{M_u}(\boldsymbol{\xi}_1(s), s)\boldsymbol{\eta}_1(s) - \mathbf{M_u}(\boldsymbol{\xi}_2(s), s)\boldsymbol{\eta}_2(s)|$ as

$$|\mathbf{M_u}(\boldsymbol{\xi}_1(s), s)(\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)) + (\mathbf{M_u}(\boldsymbol{\xi}_1(s), s) - \mathbf{M_u}(\boldsymbol{\xi}_2(s), s))\,\boldsymbol{\eta}_2(s)|.$$

Since $(\mathbf{y}_i, \mathbf{w}_i)$ solves the MR equation on $[0, \delta + h)$, inequality (2.19) applies and we have

$$\|\boldsymbol{\eta}_i\|_\infty := \sup_{\delta \le \tau < \delta + h} |\boldsymbol{\eta}_i(\tau)| \le \sup_{0 \le \tau < \delta + h} |\mathbf{w}_i(\tau)| \le \frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M}, \quad i \in \{0, 1\}.$$

Taking the supremum over $\tau \in [\delta, \delta + h)$ on both sides of (2.31) and using the above upper bound on $\|\boldsymbol{\eta}_i\|_\infty$, we get

$$\begin{aligned}\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty \le \epsilon h L_c \left[\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_\infty + \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_\infty\right] \\ + \epsilon h L_M \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_\infty \\ + \epsilon h L_c \left[\left(\frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M}\right) + 1\right] \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_\infty, \\ \le 2\epsilon h \left[3L_c + L_M + L_c \left(\frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M}\right)\right] \|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty.\end{aligned}$$

Taking $h > 0$ small enough, we get $\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty \le \frac{1}{2}\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty$, which in turn implies the uniqueness of the solution: $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_2$. The time window h can, for instance, be chosen as

$$h = \frac{1}{2} \min \left(\frac{1}{\epsilon(L_M + 1)}, \frac{1}{2\epsilon \left[3L_c + L_M + L_c \left(\frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M}\right)\right]} \right), \tag{2.32}$$

which also respects inequality (2.27). With this h, therefore, the continuation (2.23) is unique. □

Remark 2.2 *The above analysis is a contraction mapping argument. It is tempting, therefore, to use the Banach fixed point theorem (instead of the Schauder's fixed point theorem) in order to obtain the existence and uniqueness of the continuation* (2.23) *at once. The Banach fixed point theorem, however, does not apply here. This is because in proving the above contraction property, we made use of inequality* (2.19) *which applies to the mild solutions of the MR equation. As a result, it was necessary to show the existence of continuation* (2.23) *first. Otherwise, inequality* (2.19) *does not apply and the estimates used in the above contraction argument fail.*

So far we have proved the existence of a unique mild solution to the MR equation over the time interval $[0, \delta + h)$ with h given in (2.32). The steps taken in sections 2.4.1, 2.4.2 and 2.4.3 can be applied to this extended local solution to prove the existence and uniqueness of a mild solution over the time interval $[0, \delta + 2h)$. This is because the continuation window h is independent of the constants K and δ from the complete metric space $X_K^{\delta,h}$. Applying this argument repeatedly extends the mild solution of the Maxey–Riley equation from its local interval of existence and uniqueness $[0,\delta)$ to $[0,\delta+nh]$, for any $n \in \mathbb{N}$. Thus the solution can be extended uniquely to $[0,\infty)$. This proves theorem 2.4.

2.5 Slow manifold and asymptotic dynamics

Haller and Sapsis [26] proved that the Maxey–Riley equation without the Basset–Boussinesq memory term has, for small enough particles, a globally attracting slow manifold that governs its asymptotic dynamics. Based on this result they constructed by means of a series expansion a reduced-order equation describing the dynamics on the slow manifold. This equation has half the dimension of the MR equation, a fact simplifying the analysis of the asymptotic dynamics of the MR equation without memory. Haller and Sapsis subsequently used their reduced-order equation to study the asymptotic motion of inertial particles in two-dimensional steady flows, and in the unsteady wake of a cylinder. In a later paper [37], they used the reduced-order equation to study the motion of particles in a three-dimensional unsteady simulation of the hurricane Isabel of 2003.

This reduced-order equation, however, only hold if the memory term is neglected. But as already discussed, the memory term fundamentally changes the nature of the dynamics of inertial particles, and thus cannot be neglected. It would be of interest, therefore, to derive a reduced-order equation describing the asymptotic dynamics of inertial particles with memory. To do this rigorously entails showing the existence of a globally attracting slow manifold to the MR equation with memory.

It is unclear, however, that the proof of Haller and Sapsis [26] for the existence of a slow manifold for the MR equation without memory is applicable to the MR equation with memory. To see this, we will show where their proof breaks down when applied to the MR equation with memory. Consider the $\epsilon = 0$ limit of the MR equation (2.12)

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}\tau} &= 0,\\ \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} &= \mathbf{0},\\ \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} + \kappa\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} + \mathbf{w} &= \mathbf{0}.\end{aligned} \tag{2.33}$$

The system (2.33) has an $n+1$-parameter family of fixed points satisfying $\mathbf{w} = \mathbf{0}$. (Recall that n is the number of spatial dimensions. There are n degrees of freedom arising from the choice of the initial value of $\mathbf{y}(t_0) = \mathbf{y}_0$, and another one from the choice of the initial time $t = t_0$.) Formally, for any time $T > 0$, the compact invariant set

$$M_0 = \{(\mathbf{y}, t, \mathbf{w}) : \mathbf{w} = \mathbf{0}, \mathbf{y} \in \mathcal{D}, t \in [t_0 - T, t_0 + T]\}$$

is filled with fixed points of (2.33). Note that the set M_0 is compact since it is a graph over the compact domain

$$D_0 = \{(\mathbf{y}, t) : \mathbf{y} \in \mathcal{D}, t \in [t_0 - T, t_0 + T]\}.$$

The set M_0 attracts nearby trajectories at a uniform algebraic rate of $\tau^{-3/2}$, or $(t/\epsilon)^{-3/2}$ in terms of the unscaled time, since by theorem (2.2), $\mathbf{w}(\tau) \to \mathbf{0}$ as $\mathcal{O}\left(\tau^{-3/2}\right)$ for any initial condition $\mathbf{w}(t_0)$.

Had M_0 had an exponential attracting rate and had (2.33) been a system of ordinary differential equations, then we would have concluded from the results of Fenichel [38] that M_0 gives rise to a nearby locally invariant manifold for system (2.12) (that is, for $\epsilon > 0$). This would have been enough to show the existence of a globally attracting slow manifold for the MR equation with memory. This is the approach Haller and Sapsis [26] employed; they proved that the analog of the set M_0 for the MR equation without memory, that is, when the fractional derivative in (2.33) is neglected, attracts nearby trajectories at a uniform exponential rate of $\exp(-\tau)$. Since also the analog of (2.33) for the MR equation without memory is a system of ordinary differential equations, it follows by the results of Fenichel [38] that the MR equation without memory has a globally attracting slow manifold.

But M_0 has an algebraic attracting rate and (2.33) is a system of fractional differential equations. The results of Fenichel [38], therefore, are not applicable. More generally, invariant manifold results for dynamical systems (or even semi-flows, see, e.g., Bates and Jones [39]) do not apply here since, as discussed in section 2.2.3, the MR equation does not generate a dynamical system. The result for the asymptotic limit of $\mathbf{w}(\tau)$, as stated in part (iii) of theorem 2.3, is not sufficient for the existence of a globally attracting slow manifold either. Although this result guarantees that the $\mathbf{w}$ component of every trajectory converges to a state $\mathcal{O}(\epsilon)$-close to zero in forward time, it is possible that these trajectories diverge and become unbounded when run in backward time. The set of trajectories remaining bounded $\mathcal{O}(\epsilon)$-close to zero in $\mathbf{w}$ in backward time forever may be a good candidate for the globally attracting slow manifold, as all trajectories of the MR equation would then converge to this set in forward time, but this requires further work to elucidate.

Despite these difficulties, we can nonetheless conjecture that the MR equation with memory has, for small enough particles, a globally attracting slow manifold that governs its asymptotic dynamics, and construct (at a formal level) a reduced-order equation describing the dynamics on this slow manifold.

We assume that the MR equation has a globally attracting slow manifold. A necessary condition for this to hold is that the MR equation has global solutions. Hence we assume, as stated in theorem 2.4, that (H1) and (H2) hold and $\epsilon < 1/L_M$. We write the dynamics of $\mathbf{w}(\mathbf{y}, t)$ on the slow manifold by means of a formal series expansion in powers of $\epsilon^{1/2}$

$$\mathbf{w}(\mathbf{y}, t) = \sum_{j=1}^{r} \epsilon^{(j+1)/2} \mathbf{w}^{(j+1)/2}(\mathbf{y}, t) + \mathcal{O}(\epsilon^{r/2+1}). \tag{2.34}$$

We compute the functions $\mathbf{w}^{(j+1)/2}(\mathbf{y},t)$ using the invariance of the slow manifold, which allows us to differentiate (2.34) with respect to τ. We then substitute the series (2.34) in the $\mathbf{w}(\mathbf{y},t)$ equation of (2.16), equate coefficients in equal power of ϵ, and compute the functions $\mathbf{w}^{(j+1)/2}(\mathbf{y},t)$ recursively to arbitrary order starting from $j=1$. (Ferry and Balachandar [40] derived a formal series expansion for the Maxey–Riley equation without the Faxén terms up to order $j=3$.)

Since $\epsilon\tau = t - t_0$, and therefore $\frac{d}{d\tau} = \epsilon\frac{d}{dt}$, we have that

$$\begin{aligned} \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} &= \epsilon\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \\ &= \sum_{j=1}^{r-2}\epsilon^{(j+3)/2}\frac{\mathrm{d}\mathbf{w}^{(j+1)/2}}{\mathrm{d}t} + \mathcal{O}(\epsilon^{r/2+1}) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\mathbf{w}(\mathbf{y}(t(s)),t(s))}{\sqrt{\tau-s}}\,\mathrm{d}s\right) \\ &= \epsilon^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^t \frac{\mathbf{w}(\mathbf{y}(t(s)),t(s))}{\sqrt{t-s}}\,\mathrm{d}s\right) \\ &= \sum_{j=1}^{r-1}\epsilon^{(j+2)/2}\frac{\mathrm{d}^{1/2}\mathbf{w}^{(j+1)/2}}{\mathrm{d}t^{1/2}} + \mathcal{O}(\epsilon^{r/2+1}). \end{aligned} \tag{2.36}$$

Substituting (2.34), (2.35), and (2.36) in the $\mathbf{w}$ differential equation of (2.16), and writing

$$\begin{aligned} \mathbf{M}_\mathbf{u} &= \nabla\mathbf{u} + \epsilon\frac{\gamma}{6}\nabla\Delta\mathbf{u}, \\ \mathbf{B}_\mathbf{u} &= \left(\frac{3R}{2}-1\right)\left[\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}-\mathbf{g}\right] + \epsilon\left(\frac{R}{20}-\frac{1}{6}\right)\gamma\frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t}, \end{aligned}$$

we obtain

$$\begin{aligned}\sum_{j=1}^{r}\epsilon^{(j+1)/2}\mathbf{w}^{(j+1)/2} = &-\sum_{j=1}^{r-2}\epsilon^{(j+3)/2}\left[\frac{\mathrm{d}\mathbf{w}^{(j+1)/2}}{\mathrm{d}t}+\nabla\mathbf{u}\mathbf{w}^{(j+1)/2}\right]\\ &-\frac{\gamma}{6}\nabla\Delta\mathbf{u}\sum_{j=1}^{r-4}\epsilon^{(j+5)/2}\mathbf{w}^{(j+1)/2}\\ &+\epsilon\left(\frac{3R}{2}-1\right)\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}-\mathbf{g}\right)\\ &+\epsilon^2\left(\frac{R}{20}-\frac{1}{6}\right)\gamma\frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t}\\ &-\kappa\sum_{j=1}^{r-1}\epsilon^{(i+2)/2}\frac{\mathrm{d}^{1/2}\mathbf{w}^{(j+1)/2}}{\mathrm{d}t^{1/2}}+\mathcal{O}(\epsilon^{r/2+1}).\end{aligned} \tag{2.37}$$

Equating terms of equal powers, we get

$$\mathcal{O}(\epsilon):\mathbf{w}^1(\mathbf{y},t)=\left(\frac{3R}{2}-1\right)\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}-\mathbf{g}\right), \tag{2.38}$$

$$\mathcal{O}(\epsilon^{3/2}):\mathbf{w}^{3/2}(\mathbf{y},t)=-\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}^1}{\mathrm{d}t^{1/2}}, \tag{2.39}$$

$$\begin{aligned}\mathcal{O}(\epsilon^2):\mathbf{w}^2(\mathbf{y},t)=-&\left[\frac{\mathrm{D}\mathbf{w}^1}{\mathrm{D}t}+\nabla\mathbf{u}(\mathbf{y},t)\mathbf{w}^1\right.\\ &\left.-\left(\frac{R}{20}-\frac{1}{6}\right)\gamma\frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t}+\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}^{3/2}}{\mathrm{d}t^{1/2}}\right],\end{aligned} \tag{2.40}$$

$$\mathcal{O}(\epsilon^{5/2}):\mathbf{w}^{5/2}(\mathbf{y},t)=-\left[\frac{\mathrm{D}\mathbf{w}^{3/2}}{\mathrm{D}t}+\nabla\mathbf{u}\mathbf{w}^{3/2}+\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}^2}{\mathrm{d}t^{1/2}}\right], \tag{2.41}$$

and

$$\begin{aligned}\mathcal{O}(\epsilon^k):\mathbf{w}^k=-&\left[\frac{\mathrm{D}\mathbf{w}^{k-1}}{\mathrm{D}t}+\nabla\mathbf{u}\mathbf{w}^{k-1}\frac{\gamma}{6}\nabla\Delta\mathbf{u}\mathbf{w}^{k-2}+\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}^{k-1/2}}{\mathrm{d}t^{1/2}}\right],\\ &k\geq 3.\end{aligned} \tag{2.42}$$

Since $\dot{\mathbf{y}} = \mathbf{u} + \mathbf{w} + \frac{\gamma}{6}\epsilon\Delta\mathbf{u}$, which follows from rescaling back the $\mathbf{y}$ differential equation of (2.16) to the slow time t, we finally obtain the equation of motion for inertial particles on the slow manifold:

$$\begin{aligned}\dot{\mathbf{y}} = \mathbf{u}(\mathbf{y},t) + \epsilon\left[\mathbf{w}^1(\mathbf{y},t) + \frac{\gamma}{6}\Delta\mathbf{u}(\mathbf{y},t)\right] \\ + \sum_{j=2}^{r}\epsilon^{(j+1)/2}\mathbf{w}^{(j+1)/2}(\mathbf{y},t) + \mathcal{O}(\epsilon^{r/2+1}),\end{aligned} \tag{2.43}$$

where the functions $\mathbf{w}^{(j+1)/2}$ are given by (2.38), (2.39), (2.40), (2.41), and (2.42).

To order $\mathcal{O}(\epsilon)$, the equation of motion is

$$\dot{\mathbf{y}} = \mathbf{u}(\mathbf{y},t) + \epsilon\left[\left(\frac{3R}{2} - 1\right)\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{y},t) - \mathbf{g}\right) + \frac{\gamma}{6}\Delta\mathbf{u}(\mathbf{y},t)\right]. \tag{2.44}$$

This truncation agrees with the calculation Haller and Sapsis [26] and Ferry and Balachandar [40] obtained when the Faxén corrections are neglected.

To order $\mathcal{O}(\epsilon^{3/2})$, the equation of motion is

$$\begin{aligned}\dot{\mathbf{y}} = \mathbf{u}(\mathbf{y},t) + \epsilon\left[\mathbf{w}^1(\mathbf{y}(t),t) + \frac{\gamma}{6}\Delta\mathbf{u}(\mathbf{y},t)\right] \\ - \epsilon^{3/2}\kappa\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}\frac{\mathbf{w}^1(\mathbf{y}(s),s)}{\sqrt{t-s}}\,\mathrm{d}s\right)\right],\end{aligned} \tag{2.45}$$

where $\mathbf{w}^1(\mathbf{y},t)$ is given by (2.38). This is the lowest order truncation of (2.43) that incorporates memory effects.

To order $\mathcal{O}(\epsilon^2)$, the equation of motion is

$$\begin{aligned}\dot{\mathbf{y}} = \mathbf{u}(\mathbf{y},t) + \epsilon\left[\mathbf{w}^1(\mathbf{y}(t),t) + \frac{\gamma}{6}\Delta\mathbf{u}(\mathbf{y},t)\right] \\ - \epsilon^{3/2}\kappa\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}\frac{\mathbf{w}^1(\mathbf{y}(s),s)}{\sqrt{t-s}}\,\mathrm{d}s\right)\right] \\ - \epsilon^2\left[\frac{\mathrm{D}\mathbf{w}^1}{\mathrm{D}t}(\mathbf{y},t) + \nabla\mathbf{u}(\mathbf{y},t)\mathbf{w}^1(\mathbf{y},t) - \left(\frac{R}{20}-\frac{1}{6}\right)\gamma\frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t}(\mathbf{y},t)\right. \\ \left. + \kappa\frac{\mathrm{d}^{1/2}\mathbf{w}^{3/2}}{\mathrm{d}t^{1/2}}(\mathbf{y},t)\right],\end{aligned} \tag{2.46}$$

where $\mathbf{w}^1(\mathbf{y},t)$ and $\mathbf{w}^{3/2}(\mathbf{y},t)$ are given by (2.38) and (2.39). This truncation agrees with the calculation of Ferry and Balachandar [40] obtained when the Faxén corrections are neglected.

2.6 Conclusion

Motivated by the recent observations on the relevance of the memory effects on inertial particle dynamics, we have derived global existence and asymptotic decay results for the Maxey–Riley equation in the presence of the Basset–Boussinesq memory term. This memory term, a fractional derivative of order $1/2$ [3, 35], greatly complicates the analytical and numerical treatment of the equation. Although the behavior of the solutions has been well-understood in the absence of the memory term [24–26, 41], no global analytic results have been available for the full equation with memory.

We have proved that the solutions converge asymptotically to a trapping region where the particle velocity is $\mathcal{O}(\epsilon)$-close to the fluid velocity. Here, ϵ is proportional to $(a/L)^2$ where a is the particle radius and L is the characteristic length-scale of the fluid flow. This result holds for $0 < \epsilon \ll 1$ small enough which translates into $a \ll L$ (see theorem 2.3 for the exact statement of the assumption). This assumption is not restrictive since the MR equation is only valid under the very same condition $a \ll L$ [1].

We have also derived an upper envelope for the transient dynamics. This envelope exhibits an algebraic decay to the asymptotic state, hence confirming the numerical observations of [21–23] in a more general framework. We showed with an example that this envelope can be saturated and hence the upper estimates are sharp.

On neglecting the memory term, the convergence to the asymptotic limit is exponential [24–26]. The Basset–Boussinesq memory, therefore, fundamentally alters the behavior of the inertial particles and cannot be neglected. From a mathematical point of view, the memory term also fundamentally changes the structure of the equation. In the absence of memory, the Maxey–Riley equation is an ordinary differential equation that generates a dynamical system. The memory term turns the equation into a fractional differential equation that does not generate a dynamical system.

Our asymptotic results are only applicable if the Maxey–Riley equation has global solutions. Because of the particular coupling and nonlinearity of the equation, available results on fractional differential equations do not guarantee the existence and uniqueness of global solutions to the Maxey–Riley equation. To this end, we have presented the first proof of the global existence and uniqueness of mild solutions to the Maxey–Riley equation. As already pointed out by Farazmand and Haller [3], the particle velocity is not differentiable at the initial time but is continuous for all times.

Haller and Sapsis [26] constructed a reduced-order equation for the asymptotic dynamics of the MR equation without memory. This equation describes the dynamics on a globally attracting slow manifold that Haller and Sapsis showed to exist. It would be of interest to derive, on a rigorous level, a similar equation for the MR equation with memory. To do this would entail showing the existence of a slow manifold for the

MR equation with memory. But it is unclear that the proof presented by Haller and Sapsis for the existence slow manifold of the MR equation without memory is applicable to the MR equation with memory. Here, we have assumed that such a slow manifold exists for the MR equation with memory, and have constructed by means of a formal series expansion a reduced-order equation that describes the dynamics on this slow manifold. We do not know if MR equation with memory has such a slow manifold; the proof of its existence, or the lack thereof, remains an open problem.

Appendix

2.A Proof of theorem 2.2

Consider the fractional differential equation

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} + \kappa \frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} + \mathbf{w} = 0, \\ &\mathbf{w}(0) = \mathbf{w}_0. \end{aligned} \tag{2.47}$$

Let $\mathbf{W}(s) = (\mathcal{L}[\mathbf{w}])(s)$ denote the Laplace transform of $\mathbf{w}(\tau)$. Since

$$\left(\mathcal{L}\left[\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau}\right]\right)(s) = s\mathbf{W}(s) - \mathbf{w}_0$$

and

$$\left(\mathcal{L}\left[\frac{1}{\sqrt{\tau}}\right]\right)(s) = \sqrt{\frac{\pi}{s}},$$

the Laplace transform of the Riemann–Liouville derivative in (2.47) has the expression

$$\begin{aligned}\left(\mathcal{L}\left[\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}}\right]\right)(s) &= \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\int_0^\tau \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau}\frac{1}{\sqrt{\tau-\xi}}\,\mathrm{d}\xi\right]\right)(s) + \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\frac{\mathbf{w}_0}{\sqrt{\tau}}\right]\right)(s) \\ &= \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau}\right]\right)(s)\left(\mathcal{L}\left[\frac{1}{\sqrt{\tau}}\right]\right)(s) + \frac{\mathbf{w}_0}{\sqrt{s}} \\ &= (s\mathbf{W}(s) - \mathbf{w}_0)\frac{1}{\sqrt{s}} + \frac{\mathbf{w}_0}{\sqrt{s}} \\ &= \sqrt{s}\mathbf{W}(s),\end{aligned}$$

where we used the identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \frac{\mathbf{w}(s)}{\sqrt{\tau-s}}\,\mathrm{d}s = \int_0^\tau \frac{\dot{\mathbf{w}}(s)}{\sqrt{\tau-s}}\,\mathrm{d}s + \frac{\mathbf{w}(0)}{\sqrt{\tau}}.$$

Now we use the Laplace transform on (2.47) and solve for $\mathbf{W}(s)$ to get

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{s + \kappa\sqrt{s} + 1}.$$

The denominator can be factorized as

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)},$$

where

$$\lambda_\pm = \frac{\left(\kappa \pm \sqrt{\kappa^2-4}\right)}{2}.$$

Hence the general solution of (2.47) is

$$\mathbf{w}(\tau;\mathbf{w}_0) = \mathbf{w}_0\left(\mathcal{L}^{-1}\left[\frac{1}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)}\right]\right)(\tau). \tag{2.48}$$

The function $\mathbf{w}(\tau; \mathbf{w}_0)$ is proportional to the Mittag-Leffler function of order 1/2, which is defined as

$$E_{1/2}\left(-z\sqrt{\tau}\right) = e^{z^2\tau}\operatorname{erfc} z\sqrt{\tau} \tag{2.49}$$

for any complex number $z \in \mathbb{C}$ (see, e.g., [42, section 18.1]). Its Laplace transform is given by

$$\left(\mathcal{L}\left[E_{1/2}\left(-z\sqrt{\tau}\right)\right]\right)(s) = \frac{1}{\sqrt{s}\left(\sqrt{s}+z\right)}, \tag{2.50}$$

again, for any complex number $z \in \mathbb{C}$ (see, e.g., [43, equation 11.13]).

To study the behavior of $E_{1/2}\left(-z\sqrt{\tau}\right)$ as $\tau \to \infty$, we will make use of the asymptotic expansion of the complementary error function (see, e.g., [44, equation 7.1.23]):

$$\operatorname{erfc} z\sqrt{\tau} \sim \frac{e^{-z^2\tau}}{z\sqrt{\pi\tau}}\left(1 - \frac{1}{2z^2\tau} + \frac{3}{4z^4\tau^2} + \mathcal{O}\left(\frac{1}{z^6\tau^3}\right)\right). \tag{2.51}$$

On substitution in (2.49), we get

$$E_{1/2}\left(-z\sqrt{\tau}\right) \sim \frac{1}{z\sqrt{\pi\tau}}\left(1 - \frac{1}{2z^2\tau} + \frac{3}{4z^4\tau^2} + \mathcal{O}\left(\frac{1}{z^6\tau^3}\right)\right). \tag{2.52}$$

The asymptotic expansion of $\operatorname{erfc} z\sqrt{\tau}$ is valid only if $|\arg(z)| < \frac{3\pi}{4}$ (see, e.g., [44]). It also diverges for any finite value of $z\sqrt{\tau}$; its sole purpose is to give the rate of decay as $\tau \to \infty$.

The general solution will depend on whether the discriminant of $\lambda_\pm$, that is, $\kappa^2 - 4$, is positive, zero, or negative.

2.A.1 Case 1: $\kappa > 2$ ($R > 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)},$$

or, after some algebra,

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\frac{\lambda_+}{\sqrt{s}\left(\sqrt{s}+\lambda_+\right)} - \frac{\lambda_-}{\sqrt{s}\left(\sqrt{s}+\lambda_-\right)}\right].$$

Invert the two terms in the above expression with the rule (2.50) to get

$$\mathbf{w}(\tau;\mathbf{w}_0) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\lambda_+ E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) - \lambda_- E_{1/2}\left(-\lambda_-\sqrt{\tau}\right)\right]. \quad (2.53)$$

Since $\kappa - \sqrt{\kappa^2 - 4}$ is always greater than zero, we can use the asymptotic expansion (2.52) to find that in the limit $\tau \to \infty$,

$$\begin{aligned}\mathbf{w}(\tau;\mathbf{w}_0) &\sim \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\lambda_+^2\tau}\right)\right.\\ &\qquad \left. - \frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\lambda_-^2\tau}\right) + \mathcal{O}\left(\tau^{-5/2}\right)\right]\\ &\sim \frac{\mathbf{w}_0}{2\sqrt{\pi}\left(\lambda_+ - \lambda_-\right)}\left(\frac{\lambda_+^2 - \lambda_-^2}{\lambda_+^2\lambda_-^2}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right)\\ &\sim \left(\frac{\kappa\mathbf{w}_0}{2\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right),\end{aligned} \quad (2.54)$$

where we used that $\lambda_+ + \lambda_- = \kappa$ and $\lambda_+\lambda_- = 1$.

2.A.2 Case 2: $\kappa = 2$ ($R = 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s}+1\right)^2}, \quad (2.55)$$

or, after some algebra,

$$\begin{aligned}\mathbf{W}(s) &= \mathbf{w}_0\left(\frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)} - \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}\right)\\ &= \mathbf{w}_0\left(\frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)} + 2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\sqrt{s}+1}\right)\right).\end{aligned} \quad (2.56)$$

We can invert the first term in (2.56) with (2.50). The second term can be inverted by using the Laplace transforms (see, e.g., [34], equations A.27, A.28, and A.35):

$$\left(\mathcal{L}\left[\frac{1}{\sqrt{\pi\tau}} - E_{1/2}\left(-\sqrt{\tau}\right)\right]\right)(s) = \frac{1}{\sqrt{s}+1} \tag{2.57}$$

and

$$\left(\mathcal{L}\left[-\tau f(\tau)\right]\right)(s) = \frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{L}\left[f(\tau)\right])(s). \tag{2.58}$$

Thus the inverse Laplace transform of (2.55) is

$$\mathbf{w}(\tau;\mathbf{w}_0) = \mathbf{w}_0\left[E_{1/2}\left(-\sqrt{\tau}\right)(1+2\tau) - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\right]. \tag{2.59}$$

With the asymptotic expansion (2.52) we find that in the limit $\tau \to \infty$,

$$\begin{aligned}\mathbf{w}(\tau;\mathbf{w}_0) \sim \mathbf{w}_0 &\left[\frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right)\right.\\ &\left. + \frac{2\sqrt{\tau}}{\sqrt{\pi}}\left(1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right) - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\right]\\ \sim &\left(\frac{\mathbf{w}_0}{\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right).\end{aligned} \tag{2.60}$$

2.A.3 Case 3: $0 < \kappa < 2$ ($R < 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)}.$$

This is the same Laplace transform as in the case $\kappa > 2$, except that λ_+ and λ_- are now complex conjugate numbers. The inverse Laplace transform is the same as (2.53):

$$\mathbf{w}(\tau;\mathbf{w}_0) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\lambda_+ E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) - \lambda_- E_{1/2}\left(-\lambda_-\sqrt{\tau}\right)\right]. \tag{2.61}$$

The quotients

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} = \frac{1}{2}\left(1 - i\frac{\kappa}{\sqrt{4-\kappa^2}}\right)$$

and

$$-\frac{\lambda_-}{\lambda_+ - \lambda_-} = \frac{1}{2}\left(1 + i\frac{\kappa}{\sqrt{4-\kappa^2}}\right)$$

in (2.61) are also complex conjugates. Since also $\left(e^{\overline{z}}\right) = \overline{\left(e^{z}\right)}$ and $\operatorname{erfc}\overline{z} = \overline{\operatorname{erfc} z}$ for any complex number $z \in \mathbb{C}$, it follows that $E_{1/2}\left(\overline{z}\right) = \overline{E_{1/2}\left(z\right)}$. Thus,

$$\mathbf{w}(\tau;\mathbf{w}_0) = \mathbf{w}_0\left[\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right) + \overline{\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)}\right],$$

or twice the real part of $\mathbf{w}(\tau;\mathbf{w}_0)$,

$$\begin{aligned}\mathbf{w}(\tau;\mathbf{w}_0) &= 2\mathbf{w}_0 \operatorname{Re}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}(-\lambda_+\sqrt{\tau})\right)\\ &= 2\mathbf{w}_0\left[\operatorname{Re}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)\operatorname{Re}\left(E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)\right. \\ &\quad \left. + \operatorname{Im}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)\operatorname{Im}\left(E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)\right].\end{aligned} \tag{2.62}$$

It is possible to further simplify (2.61). The Mittag-Leffler function $E_{1/2}\left(-z\right)$ can be written as (see, e.g., [45, section 7.19])

$$E_{1/2}\left(-z\right) = \sqrt{\frac{4t}{\pi}}\left[U\left(x,t\right) + iV\left(x,t\right)\right], \tag{2.63}$$

where

$$U\left(x,t\right) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{e^{-(x+s)^2/(4t)}}{1+s^2}\,\mathrm{d}s, \tag{2.64}$$

$$V\left(x,t\right) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{se^{-(x+s)^2/(4t)}}{1+s^2}\,\mathrm{d}s, \tag{2.65}$$

with $x \in \mathbb{R}$, $t > 0$, and $z = \frac{1-ix}{2\sqrt{t}}$. The functions $U(x,t)$ and $V(x,t)$ are known as the Voigt functions (see, e.g., [45, section 7.19]). If we set

$$z = \frac{1-ix}{2\sqrt{t}} = \lambda_+\sqrt{\tau} = \left(\frac{\kappa}{2} + i\frac{\sqrt{4-\kappa^2}}{2}\right)\sqrt{\tau},$$

then we can solve for x and t to get

$$t = \frac{1}{\kappa^2\tau}$$

and

$$x = -\frac{\sqrt{4-\kappa^2}}{\kappa}.$$

Thus

$$\begin{aligned} E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) = \frac{2}{\kappa\sqrt{\pi\tau}}\Bigg[&U\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right) \\ &- iV\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right)\Bigg]. \end{aligned} \tag{2.66}$$

Hence (2.62) can be written as

$$\begin{aligned} \mathbf{w}(\tau;\mathbf{w}_0) = \frac{2\mathbf{w}_0}{\kappa\sqrt{\pi\tau}}\Bigg[&U\left(-\frac{\sqrt{4-\kappa^2}}{a}, \frac{1}{\kappa^2\tau}\right) \\ &- \frac{\kappa}{\sqrt{4-\kappa^2}}V\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right)\Bigg]. \end{aligned} \tag{2.67}$$

For the asymptotic behavior of $\mathbf{w}(\tau;\mathbf{w}_0)$ as $\tau \to \infty$, we can repeat the steps as in the case $\kappa > 2$ and get

$$\mathbf{w}(\tau;\mathbf{w}_0) \sim \left(\frac{\kappa\mathbf{w}_0}{2\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right). \tag{2.68}$$

This asymptotic expansion, however, is justified only if $|\arg(\lambda_+\sqrt{\tau})|$ and $|\arg(\lambda_+\sqrt{\tau})|$ are smaller than $\frac{3\pi}{4}$. Because $\lambda_\pm = \left(\kappa \pm i\sqrt{4-\kappa^2}\right)/2$, we see that this will be the case whenever $\kappa > 0$ since then $0 < \arg\left(\lambda_+\sqrt{\tau}\right) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg\left(\lambda_-\sqrt{\tau}\right) < 0$. (To see this, note that the two complex numbers λ_+ and λ_- lie to the right of the imaginary axis, so that the argument cannot be greater than $|\pi/2|$.) As $\kappa = \sqrt{9R/2}$ and $R > 0$, the required condition $\kappa > 0$ is always satisfied.

2.B Proof of theorem 2.3

We will use the following Gronwall-type inequality.

Lemma 2.3 (Chu & Metcalf [46]) *Let the functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}$ be continuous and the function $K(\tau, s)$ be continuous and nonnegative for $0 \le s \le \tau$. If*

$$\alpha(\tau) \le \beta(\tau) + \int_0^\tau K(\tau, s)\alpha(s)\,\mathrm{d}s,$$

then

$$\alpha(\tau) \le \beta(\tau) + \int_0^\tau H(\tau, s)\beta(s)\,\mathrm{d}s,$$

where $H(\tau, s) = \sum_{j=1}^{\infty} K_j(\tau, s)$, $K_1(\tau, s) = K(\tau, s)$ and

$$K_j(\tau, s) = \int_s^\tau K_{j-1}(\tau, \xi)K(\xi, s)\,\mathrm{d}\xi, \quad j \ge 2.$$

Corollary 2.1 *If $K(\tau, s) = k(\tau - s)$, then in fact $K_j(\tau, s) = k_j(\tau - s)$, where*

$$k_j(\tau) = (k * k * \cdots * k)(\tau),$$

where the convolution is j-fold. As a result, $H(\tau, s) = h(\tau - s)$, where

$$h(\tau) = \sum_{j=1}^{\infty} k_j(\tau).$$

Proof We prove $K_2(\tau, s) = k * k(\tau - s)$. The rest follows similarly by induction. We have

$$\begin{aligned} K_2(\tau,s) &:= \int_s^\tau K(\tau,\xi)K(\xi,s)\,\mathrm{d}\xi \\ &= \int_s^\tau k(\tau-\xi)k(\xi-s)\,\mathrm{d}\xi \\ &= \int_0^{\tau-s} k(\tau-s-\eta)k(\eta)\,\mathrm{d}\eta \\ &= k*k(\tau-s) =: k_2(\tau-s), \end{aligned}$$

where we used the change of variable $\eta = \xi - s$. □

Proof (theorem 2.3) It follows from the integral equation (2.17) that

$$\begin{aligned} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \le \psi(\tau)|\mathbf{w}_0| + \epsilon L_B\left(1-\phi(\tau)\right) \\ + \epsilon L_M \int_0^\tau \psi(\tau-s)|\mathbf{w}(s;\mathbf{y}_0,\mathbf{w}_0)|\,\mathrm{d}s, \end{aligned} \tag{2.69}$$

where $\tau \in [0,\delta)$. Using lemma 2.3 with $\alpha(\tau) = |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|$, $\beta(\tau) = \psi(\tau)|\mathbf{w}_0| + \epsilon L_B\left(1-\phi(\tau)\right)$ and $K(\tau,s) = \epsilon L_M \psi(\tau-s)$, we get

$$\begin{aligned} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| &\le \psi(\tau)|\mathbf{w}_0| + \epsilon L_B\left(1-\phi(\tau)\right) \\ &\quad + \int_0^\tau h(\tau-s)\left[\psi(s)|\mathbf{w}_0| + \epsilon L_B\left(1-\phi(\tau)\right)\right]\,\mathrm{d}s \\ &= \left[\psi(\tau) + \int_0^\tau h(\tau-s)\psi(s)\,\mathrm{d}s\right]|\mathbf{w}_0| + \epsilon L_B\left(1-\phi(\tau)\right) \\ &\quad + \epsilon L_B \int_0^\tau h(\tau-s)\left(1-\phi(s)\right)\,\mathrm{d}s, \end{aligned} \tag{2.70}$$

where $h(\tau;\epsilon) = \sum_{j=1}^\infty k_j(\tau)$ with $k_1 = \epsilon L_M \psi(\tau)$ and $k_j = k_{j-1} * k_1$. Induction on j leads to the expression

$$k_j = (\epsilon L_M)^j \psi^{*j}.$$

Therefore we have the identity

$$\begin{aligned}
\psi(\tau) + \int_0^\tau h(\tau-s)\psi(s)\,\mathrm{d}s &= \frac{k_1(\tau)}{\epsilon L_M} + \int_0^\tau \sum_{j=1}^\infty k_j(\tau-s)\frac{k_1(s)}{\epsilon L_M}\,\mathrm{d}s \\
&= \frac{k_1(\tau)}{\epsilon L_M} + \frac{1}{\epsilon L_M}\sum_{j=1}^\infty \int_0^\tau k_j(\tau-s)k_1(s)\,\mathrm{d}s \\
&= \frac{k_1(\tau)}{\epsilon L_M} + \frac{1}{\epsilon L_M}\sum_{j=1}^\infty k_{j+1}(\tau) \\
&= \frac{1}{\epsilon L_M}\sum_{j=1}^\infty k_j(\tau) \\
&= \frac{1}{\epsilon L_M} h(\tau),
\end{aligned}$$

where we omitted the dependence of h on the parameter ϵ for notational simplicity.

This shows that

$$\begin{aligned}
|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{|\mathbf{w}_0|}{\epsilon L_M} h(\tau) + \epsilon L_B\,(1-\phi(\tau)) \\
+ \epsilon L_B \int_0^\tau h(\tau-s)\,(1-\phi(s))\,\mathrm{d}s.
\end{aligned} \tag{2.71}$$

Since $0 \leq \phi(\tau) \leq 1$, we have that $(1-\phi(\tau)) \leq 1$ and therefore the inequality can be further simplified to

$$|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{|\mathbf{w}_0|}{\epsilon L_M} h(\tau) + \epsilon L_B\,[1-\phi(\tau)] + \epsilon L_B \int_0^\tau h(s)\,\mathrm{d}s. \tag{2.72}$$

So far we have assumed that the series $\sum_{j=1}^\infty k_j = \sum_{j=1}^\infty (\epsilon L_M)^j \psi^{*j}$ converges uniformly to a limit h. To prove this, we first show that for any j and $\tau \geq 0$, $0 \leq \psi^{*j}(\tau) \leq 1$. For $j = 1$, this property holds since $0 \leq \psi \leq 1$. For $j = 2$ we have

$$0 \leq \psi^{*2}(\tau) := \int_0^\tau \psi(\tau-s)\psi(s)\,\mathrm{d}s \leq \int_0^\tau \psi(s)\,\mathrm{d}s = 1-\phi(\tau) \leq 1.$$

By induction on j, we get $0 \leq \psi^{*j}(\tau) \leq 1$. As a result, $(\epsilon L_M)^j \psi^{*j} \leq (\epsilon L_M)^j$. Since $\epsilon L_M < 1$, the series $\sum_{j=1}^{\infty} (\epsilon L_M)^j$ converges. It follows that

$$|h(\tau)| \leq \sum_{j=1}^{\infty} (\epsilon L_M)^j = \frac{\epsilon L_M}{1 - \epsilon L_M} \tag{2.73}$$

by summing up the geometric series. By the dominated convergence theorem, the sequence $\sum_{j=1}^{n} (\epsilon L_M)^j \psi^{*j}$ converges uniformly to a function h as $n \to \infty$. Since the series $\sum_{j=1}^{n} (\epsilon L_M)^j \psi^{*j}$ is continuous for any n, so is the limiting function h. This shows that $h : [0, \infty) \to \mathbb{R}$ is continuous and $h \geq 0$.

Now, observe that

$$\begin{aligned} \int_0^{\tau} h(\xi)\, \mathrm{d}\xi &= \int_0^{\tau} \sum_{j=1}^{\infty} (\epsilon L_M)^j \psi^{*j}(\xi)\, \mathrm{d}\xi \\ &= \sum_{j=1}^{\infty} (\epsilon L_M)^j \int_0^{\tau} \psi^{*j}(\xi)\, \mathrm{d}\xi \\ &\leq \sum_{j=1}^{\infty} (\epsilon L_M)^j = \frac{\epsilon L_M}{1 - \epsilon L_M}, \end{aligned} \tag{2.74}$$

where we used the uniform convergence of the series and the fact that for any j,

$$\begin{aligned} 0 \leq \int_0^{\tau} \psi^{*j}(\xi)\, \mathrm{d}\xi &\leq \left(\int_0^{\tau} \psi^{*(j-1)}(\xi)\, \mathrm{d}\xi \right) \left(\int_0^{\tau} \psi(\xi)\, \mathrm{d}\xi \right) \\ &\leq \cdots \leq \left(\int_0^{\tau} \psi(\xi)\, \mathrm{d}\xi \right)^j = (1 - \phi(\tau))^j \leq 1, \end{aligned}$$

by repeated application of Young's inequality for convolutions. This also shows that $h(\tau) \to 0$ as $\tau \to \infty$, since $|h|_1 < \infty$ and h is uniformly continuous.

Using inequality (2.74) in (2.72) and the definition of h, we get

$$|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{|\mathbf{w}_0|}{\epsilon L_M} h(\tau) + \epsilon L_B \left(1-\phi(\tau)\right) + \frac{\epsilon^2 L_M L_B}{1-\epsilon L_M} \tag{2.75}$$

$$= |\mathbf{w}_0| \left[\sum_{j=1}^{\infty} (\epsilon L_M)^{j-1} \psi^{*j}(\tau) \right]$$

$$+ \epsilon L_B \left(1-\phi(\tau)\right) + \frac{\epsilon^2 L_M L_B}{1-\epsilon L_M}. \tag{2.76}$$

This proves part (i) of the theorem.

Taking the sup of $|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|$ over $[0,\delta)$, we get

$$\sup_{0\leq\tau<\delta} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{|\mathbf{w}_0| + \epsilon L_B}{1-\epsilon L_M}, \tag{2.77}$$

which proves part (ii) of theorem 2.3.

If $\delta = \infty$, then we can take the limitsup of $|\mathbf{w}|$. Using inequality (2.75) we get the asymptotic estimate

$$\limsup_{\tau\to\infty} |\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)| \leq \frac{\epsilon L_B}{1-\epsilon L_M}, \tag{2.78}$$

which proves part (iii) of theorem 2.3. Here, we used the fact that $\lim_{\tau\to\infty} h(\tau) = 0$ and $\lim_{\tau\to\infty} \phi(\tau) = 0$. □

2.C Proof of lemma 2.1

Let $\tau_1, \tau_2 \in [0, \delta)$. Bound $|\mathbf{z}_{loc}(\tau_2) - \mathbf{z}_{loc}(\tau_1)|$ by

$$\begin{aligned}|\mathbf{z}_{loc}(\tau_2) - \mathbf{z}_{loc}(\tau_1)| &\leq |\mathbf{y}_{loc}(\tau_2) - \mathbf{y}_{loc}(\tau_1)| + |\mathbf{w}_{loc}(\tau_2) - \mathbf{w}_{loc}(\tau_1)| \\ &\leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| \\ &\quad + \epsilon \int_{\tau_1}^{\tau_2} |\mathbf{w}_{loc}(s)| + |\mathbf{A}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)| \, ds \\ &\quad + \epsilon \int_{\tau_1}^{\tau_2} \psi(\tau_2 - s)|\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)||\mathbf{w}_{loc}(s)| \, ds \\ &\quad + \epsilon \int_{\tau_1}^{\tau_2} \psi(\tau_2 - s)|\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)| \, ds \\ &\quad + \epsilon \int_0^{\tau_1} |\psi(\tau_2 - s) - \psi(\tau_1 - s)||\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)||\mathbf{w}_{loc}(s)| \, ds \\ &\quad + \epsilon \int_0^{\tau_1} |\psi(\tau_2 - s) - \psi(\tau_1 - s)||\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)| \, ds.\end{aligned}$$

Without loss of generality, suppose $\tau_1 \leq \tau_2$, so that $|\psi(\tau_2 - s) - \psi(\tau_1 - s)| = \psi(\tau_2 - s) - \psi(\tau_1 - s)$. Taking the infinity norm over $[0, \delta)$ to bound $\|\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\|_\infty$, $\|\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\|_\infty$, $\|\mathbf{A}_{\mathbf{u}}(\mathbf{y}_{loc}(s), s)\|_\infty$, and $|\mathbf{w}_{loc}(s)|$ by theorem 2.3, we get

$$\begin{aligned}|\mathbf{z}_{loc}(\tau_2) - \mathbf{z}_{loc}(\tau_1)| &\leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| + \epsilon \left(L_A + \frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M} \right) |\tau_2 - \tau_1| \\ &\quad + \epsilon \left[\frac{L_M |\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right] |\tau_2 - \tau_1| \\ &\quad + \epsilon \left[\frac{L_M |\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right] \int_0^{\tau_1} \psi(\tau_1 - s) - \psi(\tau_2 - s) \, ds.\end{aligned}$$

By the results of theorem 2.2, $\psi(\tau_1 - s) - \psi(\tau_2 - s) = \phi'(\tau_2 - s) - \phi'(\tau_1 - s) \geq 0$. Integrate and rearrange to get

$$|\mathbf{z}_{loc}(\tau_2) - \mathbf{z}_{loc}(\tau_1)| \leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| + \epsilon\left(L_A + \frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M}\right)|\tau_2 - \tau_1|$$
$$+ \epsilon\left[\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M}\right](|\tau_2 - \tau_1| + \phi(\tau_2) - \phi(\tau_1))$$
$$+ \epsilon\left[\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M}\right](\phi(0) - \phi(\tau_2 - \tau_1)).$$

Since both ψ and ϕ are uniformly continuous over $[0, \infty)$ by theorem 2.2, each of $|\psi(\tau_2) - \psi(\tau_1)|$, $|\phi(\tau_2) - \phi(\tau_1)|$, and $|\phi(0) - \phi(\tau_2 - \tau_1)|$ converges to zero as $|\tau_2 - \tau_1|$ converges to zero. Hence $|\mathbf{z}_{loc}(\tau_2) - \mathbf{z}_{loc}(\tau_1)| \to 0$ as $\tau_1, \tau_2 \to \delta_-$.

Now, if we take a sequence $\{t_n\}$, $t_n \in [0, \delta)$, such that $\lim_{n\to\infty} t_n \to \delta$, then it follows that $\{\mathbf{z}_{loc}(t_n)\}$ is a Cauchy sequence. The sequence is convergent in $\mathbb{R}^{2n}$ since $\mathbb{R}^{2n}$ is a complete metric space. The limit is given by the integral equation (2.16) evaluated at $\tau = \delta$:

$$\mathbf{z}_{loc}(\delta)$$
$$= \begin{pmatrix} \mathbf{y}_0 + \epsilon \int_0^\delta \mathbf{w}_{loc}(s) + \mathbf{A}_\mathbf{u}(\mathbf{y}_{loc}(s), s) \, \mathrm{d}s \\ \psi(\delta)\mathbf{w}_0 + \epsilon \int_0^\delta \psi(\tau - s)\left[-\mathbf{M}_\mathbf{u}(\mathbf{y}_{loc}(s), s)\mathbf{w}_{loc}(s) + \mathbf{B}_\mathbf{u}(\mathbf{y}_{loc}(s), s)\right] \, \mathrm{d}s \end{pmatrix}.$$

This proves lemma 2.1.

2.D Proof of lemma 2.2

Let $\mathbf{\Phi} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in X_K^{\delta,h}$ and $\tau_1, \tau_2 \in [\delta, \delta+h)$. Bound $|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)|$ by

$$\begin{aligned}
|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \leq |\mathbf{\Phi}_0(\tau_2) - \mathbf{\Phi}_0(\tau_1)| &+ \epsilon \int_{\tau_1}^{\tau_2} |\boldsymbol{\eta}(s)| + |\mathbf{A_u}(\boldsymbol{\xi}(s), s)| \, \mathrm{d}s \\
&+ \epsilon \int_{\tau_1}^{\tau_2} \psi(\tau_2 - s) \left[|\mathbf{M_u}(\boldsymbol{\xi}(s), s)||\boldsymbol{\eta}(s)| + |\mathbf{B_u}(\boldsymbol{\xi}(s), s)| \right] \, \mathrm{d}s \\
&+ \epsilon \int_{\delta}^{\tau_1} (\psi(\tau_2 - s) - \psi(\tau_1 - s)) \, |\mathbf{M_u}(\boldsymbol{\xi}(s), s)||\boldsymbol{\eta}(s)| \mathrm{d}s \\
&+ \epsilon \int_{\delta}^{\tau_1} (\psi(\tau_2 - s) - \psi(\tau_1 - s)) \, |\mathbf{B_u}(\boldsymbol{\xi}(s), s)| \, \mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
|\mathbf{\Phi}_0(\tau_2) - \mathbf{\Phi}_0(\tau_1)| \leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| & \\
&+ \epsilon \int_0^{\delta} |\psi(\tau_2 - s) - \psi(\tau_1 - s)||\mathbf{M_u}(\mathbf{y}_{loc}(s), s)||\mathbf{w}_{loc}(s)| \, \mathrm{d}s \\
&+ \epsilon \int_0^{\delta} |\psi(\tau_2 - s) - \psi(\tau_1 - s)||\mathbf{B_u}(\mathbf{y}_{loc}(s), s)| \, \mathrm{d}s.
\end{aligned}$$

Without loss of generality, suppose $\tau_1 \leq \tau_2$, so that $|\psi(\tau_2 - s) - \psi(\tau_1 - s)| = \psi(\tau_2 - s) - \psi(\tau_1 - s)$. Taking the infinity norm over $[\delta, \delta + h)$ to bound $\|\mathbf{M_u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\mathbf{B_u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\mathbf{A_u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\boldsymbol{\eta}(s)\|_\infty$, and $|\mathbf{w}_{loc}(s)|$ by inequality (2.19), we get

$$\begin{aligned}
|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| & \\
&+ \epsilon \left(\frac{L_M |\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right) \int_0^{\delta} \psi(\tau_1 - s) - \psi(\tau_2 - s) \, \mathrm{d}s \\
&+ \epsilon (K + L_A)|\tau_2 - \tau_1| + \epsilon \left(L_M K + L_B \right) |\tau_2 - \tau_1| \\
&+ \epsilon \left(L_M K + L_B \right) \int_{\delta}^{\tau_1} \psi(\tau_1 - s) - \psi(\tau_2 - s) \, \mathrm{d}s.
\end{aligned}$$

By the results of theorem 2.2, $\psi(\tau_1 - s) - \psi(\tau_2 - s) = \phi'(\tau_2 - s) - \phi'(\tau_1 - s) \geq 0$. Finally, integrate and rearrange to get

$$\begin{aligned}
|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| \\
+ \epsilon \left(\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right) (\phi(\tau_1 - \delta) - \phi(\tau_2 - \delta)) \\
+ \epsilon \left(\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right) (\phi(\tau_2) - \phi(\tau_1)) \\
+ \epsilon(K + L_A)|\tau_2 - \tau_1| + \epsilon \left(L_M K + L_B\right) |\tau_2 - \tau_1| \\
+ \epsilon \left(L_M K + L_B\right) (\phi(0) - \phi(\tau_2 - \tau_1)) \\
+ \epsilon \left(L_M K + L_B\right) (\phi(\tau_2 - \delta) - \phi(\tau_1 - \delta)) .
\end{aligned}$$

Since both ψ and ϕ are uniformly continuous over $[0, \infty)$ by theorem 2.2, each of $|\phi(\tau_1 - \delta) - \phi(\tau_2 - \delta)|$, $|\psi(\tau_2) - \psi(\tau_1)|$, $|\phi(0) - \phi(\tau_2 - \tau_1)|$, and $|\phi(\tau_2) - \phi(\tau_1)|$ converges to zero as $|\tau_2 - \tau_1|$ converges to zero. Hence $|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \to 0$ as $|\tau_2 - \tau_1| \to 0$. This shows that $\mathbf{F}$ maps $X_K^{\delta,h}$ to a family of uniformly equicontinuous functions in $C([\delta, \delta + h)\,; \mathbb{R}^{2n})$. This proves lemma 2.2.

Chapter 3

Explicit form of spatially linear Navier-Stokes velocity fields

3.1 Introduction

In this chapter, we answer the following simple question: For what matrices $\mathbf{A}(t)$ and vectors $\mathbf{f}(t)$ does the linear velocity field

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) \tag{3.1}$$

solve the Navier–Stokes equations?

Apart from the general existence conditions of Majda [4], Majda and Bertozzi [5], and Craik and Criminale [6], a treatment of solutions of the form (3.1) has been lacking. Extensive reviews of exact Navier–Stokes solutions also omit a discussion of this general class of velocity fields, although they list several specific (mostly steady) spatially linear solutions for concrete physical settings [47–51].

The spatially linear velocity field (3.1) can still have arbitrary temporal complexity, including highly unsteady or even chaotic time dependence on $\mathbf{A}(t)$ and $\mathbf{f}(t)$. This makes spatially linear Navier–Stokes flows ideal tools for testing the ability of vortex and coherent structure definitions to cope with unsteady flows in a dynamically consistent setting. In addition, spatially linear Navier–Stokes solutions are also interesting as numerical benchmarks or exact solutions to specific boundary conditions. Known examples of spatially linear, steady Navier–Stokes solutions include jet-, strain-, and rotation-type flows [5].

Available unsteady linear solutions are less frequent. Notable exceptions are the rotating jet flows of Majda and Bertozzi [5] and Craik and Allen [52], and the recent linear velocity family identified by Dayal and James [53] as a universal solution to any fluid equation with zero body force. We illustrate the general conditions on these two specific examples, and construct further examples of linear unsteady Navier–Stokes velocity fields.

3.2 Problem statement and prior results

We consider general linear velocity fields of the form (3.1), where $\mathbf{A} : [t_0, t_1) \to \mathbb{R}^{n\times n}$ is a time-dependent matrix and $\mathbf{f} : [t_0, t_1) \to \mathbb{R}^n$ is a time-dependent vector. The time interval $[t_0, t_1)$ is possibly infinitely long, and the spatial dimensions are either $n = 2$ or $n = 3$. We seek necessary and sufficient conditions on $\mathbf{A}(t)$ and $\mathbf{f}(t)$ so that the linear velocity field (3.1) solves the Navier–Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p\,(\mathbf{x}, t) + \nu \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0, \tag{3.2}$$

for an appropriate pressure function $p(\mathbf{x}, t)$. Here, ρ denotes the fluid density and ν is the kinematic viscosity.

To our knowledge, the only available treatment of spatially linear solutions (3.1) are from Majda [4] (later reprinted in [5]) and Craik and Criminale [6], who independently derived general existence condition for such flows to solve the Navier–Stokes equation. In particular, Majda showed that if $\mathbf{D}(t)$ is a 3×3 traceless symmetric matrix, and the vorticity $\boldsymbol{\omega}(\mathbf{x},t) = \nabla \times \mathbf{u}(\mathbf{x},t)$ solves the ordinary differential equation

$$\begin{aligned} \dot{\boldsymbol{\omega}} &= \mathbf{D}(t)\boldsymbol{\omega}, \\ \boldsymbol{\omega}(t_0) &= \boldsymbol{\omega}_0(\mathbf{x}), \end{aligned} \tag{3.3}$$

with initial condition $\boldsymbol{\omega}_0(\mathbf{x})$, then the linear velocity field

$$\mathbf{u}(\mathbf{x},t) = \mathbf{D}(t)\,\mathbf{x} + \frac{1}{2}\boldsymbol{\omega}(t) \times \mathbf{x} \tag{3.4}$$

solves the Navier–Stokes equation (3.2).

All linear velocity fields (3.1) (with $\mathbf{f}(t) = \mathbf{0}$) solving the Navier–Stokes equation must, therefore, satisfy conditions (3.3)-(3.4). It is, however, not immediate from these conditions what the exact form of $\mathbf{A}(t)$ will be. In addition, it is not immediately clear without further calculations whether a given velocity field (3.1) satisfies conditions (3.3) and (3.4).

Below we address these questions by finding equivalent formulations of conditions (3.3)-(3.4) that reveal the most general admissible form of $\mathbf{A}(t)$.

3.3 Summary of results

First, we show that the linear velocity field (3.1) solves the Navier–Stokes equation (3.2) if and only if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ in (3.1) satisfy three conditions:

(I) $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuously differentiable functions of time,

(II) $\mathrm{Tr}(\mathbf{A}(t)) = 0$,

(III) $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ is symmetric.

Under these conditions, the velocity field (3.1) is an exact Navier–Stokes solution for all values of ρ and ν. When of interest, the associated pressure field can also be computed directly as

$$p(\mathbf{x},t) = \rho\left[\left(\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)\right)\mathbf{x} + \mathbf{A}(t)\,\mathbf{f}(t) + \dot{\mathbf{f}}(t)\right]\cdot\mathbf{x}. \tag{3.5}$$

Conditions (I)-(III) have been derived before by Craik and Criminale [6] in the context of studying the evolution of wavelike disturbances in shear flows.

For two-dimensional flows, conditions (I)-(III) turn out to imply that

$$\mathbf{A}(t) = \begin{pmatrix} a(t) & b(t)+K \\ b(t)-K & -a(t) \end{pmatrix}, \tag{3.6}$$

where $a(t)$ and $b(t)$ are arbitrary, continuously differentiable functions of time, and $K \in \mathbb{R}$ is constant. Any choice of such $a(t)$ and $b(t)$ gives an exact Navier–Stokes solution for any continuously differentiable $\mathbf{f}(t)$.

For three-dimensional flows, conditions (I)-(III) can be shown to imply that

$$\mathbf{A}(t) = \mathbf{D}(t) + \mathbf{M}(t)\,\mathbf{W}_0\mathbf{M}^T(t), \tag{3.7}$$

where $\mathbf{D}(t)$ is an arbitrary continuously differentiable traceless symmetric matrix, $\mathbf{W}_0$ is an arbitrary constant skew-symmetric matrix, and $\mathbf{M}(t)$ is the solution of the matrix initial-value problem

$$\begin{aligned} \dot{\mathbf{M}} &= -\mathbf{D}(t)\,\mathbf{M}, \\ \mathbf{M}(t_0) &= \mathbf{I}. \end{aligned} \tag{3.8}$$

The explicit form of two- and three-dimensional flows, to our knowledge, is novel and has not appeared elsewhere.

All these results follow from elementary arguments that we spell out in detail below.

3.4 Necessary and sufficient conditions for spatially linear solutions

Our main result can be stated formally as follows:

Theorem 3.1 *The linear velocity field* (3.1) *solves the Navier–Stokes equation* (3.2) *if and only if* $\mathbf{A}(t)$ *and* $\mathbf{f}(t)$ *satisfy conditions (I)-(III) of section* 3.3*. Under these conditions, the pressure field associated with* (3.1) *is given by equation* (3.5)*.*

Proof Substitution of the linear velocity field (3.1) into the Navier–Stokes equation (3.2) gives

$$\left(\dot{\mathbf{A}} + \mathbf{A}^2\right)\mathbf{x} + \mathbf{A}\mathbf{f} + \dot{\mathbf{f}} = -\frac{1}{\rho}\nabla p, \qquad \nabla\cdot(\mathbf{A}\mathbf{x}) = 0, \tag{3.9}$$

where the viscosity term vanishes because of the linearity of $\mathbf{u}$ in $\mathbf{x}$. All terms in (3.9) are well-defined if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuously differentiable functions of time, as assumed in condition (I). We now take the spatial gradient on both sides of the first equation in (3.9) to obtain

$$\dot{\mathbf{A}}(t) + \mathbf{A}^2(t) = -\frac{1}{\rho}\nabla^2 p\,(\mathbf{x}, t)\,. \tag{3.10}$$

The right-hand side of (3.10) is a constant times the Hessian of the pressure and is therefore a symmetric matrix. Hence, a pressure function $p(\mathbf{x}, t)$ exists if and only if $\dot{\mathbf{A}}(t) + \mathbf{A}^2(t)$ is symmetric, which proves condition (III). Under this condition, we fix time and take the line integral of both sides of the first equation (3.9) along an arbitrary spatial curve $\mathbf{x}(s) \in \mathbb{R}^n$ with unit tangent vectors $\mathbf{x}'(s)$. Discarding the constants arising in the integration, we obtain the pressure function (3.5). Finally, we observe that

$$\nabla\cdot(\mathbf{A}(t)\mathbf{x}) = \mathrm{Tr}\,(\mathbf{A}\,(t))$$

therefore the second equation in (3.9) is equivalent to condition (II) of the theorem. □

Remark 3.1 *We can decompose* $\mathbf{A}(t)$ *into*

$$\mathbf{A}(t) = \mathbf{D}(t) + \mathbf{W}(t), \tag{3.11}$$

where the rate-of-strain tensor $\mathbf{D}(t) = \frac{1}{2}\left(\mathbf{A}(t) + \mathbf{A}^T(t)\right)$ *is symmetric and the spin tensor* $\mathbf{W}(t) = \frac{1}{2}\left(\mathbf{A}(t) - \mathbf{A}^T(t)\right)$ *is skew-symmetric. By a simple calculation,* $\dot{\mathbf{A}} + \mathbf{A}^2(t)$ *is symmetric precisely when* $\dot{\mathbf{D}} + \dot{\mathbf{W}} + \mathbf{D}^2(t) + \mathbf{W}^2(t) + \mathbf{D}(t)\mathbf{W}(t) + \mathbf{W}(t)\mathbf{D}(t)$ *is symmetric. This is the case for two-dimensional flows if and only if*

$$\dot{\mathbf{W}} = \mathbf{0} \tag{3.12}$$

and for three-dimensional flows if and only if

$$\dot{\mathbf{W}} + \mathbf{D}(t)\mathbf{W}(t) + \mathbf{W}(t)\mathbf{D}(t) = \mathbf{0}. \tag{3.13}$$

We will use these equivalent conditions to describe the explicit form of two-dimensional and three-dimensional spatially linear Navier–Stokes velocity fields more closely.

Example 3.1 *Dayal and James [53] showed that the family of unsteady velocity fields*

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}_0 \left[\mathbf{I} + (t - t_0)\,\mathbf{A}_0\right]^{-1} \mathbf{x} \tag{3.14}$$

with the constant matrix $\mathbf{A}_0$ *and the identity matrix* $\mathbf{I}$*, solves the Navier–Stokes equations precisely when*

$$\mathbf{A}_0 \in \left\{ \begin{pmatrix} 0 & 0 & \kappa \\ \gamma & 0 & \delta \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ \delta & \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & \delta & \gamma \\ 0 & 0 & 0 \\ 0 & \kappa & 0 \end{pmatrix} \right\},$$

with arbitrary real constants δ*,* γ*, and* κ*. The coefficient matrix*

$$\mathbf{A}(t) = \mathbf{A}_0 \left[\mathbf{I} + (t - t_0)\,\mathbf{A}_0\right]^{-1}$$

in (3.14) *is, therefore, one of the three forms*

$$\begin{pmatrix} 0 & 0 & \kappa \\ \gamma & 0 & \delta - \gamma\kappa\,(t - t_0) \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ \delta - \gamma\kappa\,(t - t_0) & \gamma & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \delta - \gamma\kappa\,(t - t_0) & \gamma \\ 0 & 0 & 0 \\ 0 & \kappa & 0 \end{pmatrix}.$$

These matrices satisfy all three conditions of theorem 3.1. Indeed, all three matrices are continuously differentiable and have zero trace, hence satisfy conditions (I) and (II). Furthermore, $\dot{\mathbf{A}} + \mathbf{A}^2(t) = \mathbf{0}$*, and hence condition (III) is also satisfied.*

3.5 Two-dimensional spatially linear Navier–Stokes solutions

The three conditions of theorem 3.1 give the following characterization for two-dimensional spatially linear solutions.

Proposition 3.1 *The two-dimensional linear velocity field* (3.1) *solves the Navier–Stokes equation* (3.2) *if and only if*

$$\mathbf{A}\,(t) = \mathbf{D}\,(t) + \mathbf{W}_0, \tag{3.15}$$

where $\mathbf{D}\,(t) \in \mathbb{R}^{2\times 2}$ *is an arbitrary continuously differentiable traceless symmetric matrix, and* $\mathbf{W}_0 \in \mathbb{R}^{2\times 2}$ *is an arbitrary constant skew-symmetric matrix. Equivalently,* $\mathbf{A}\,(t)$ *has the specific form* (3.6)*.*

Proof Applying the three conditions of theorem 3.1 to the symmetric and skew-symmetric components $\mathbf{D}(t)$ and $\mathbf{W}(t)$ of $\mathbf{A}(t)$ (see remark 3.1), it follows that $\mathbf{D}(t)$ must be a continuously differentiable traceless symmetric matrix and, by equation (3.12), $\dot{\mathbf{W}} = \mathbf{0}$ must hold. The latter yields $\mathbf{W}(t) = \mathbf{W}_0$, a constant skew-symmetric matrix. □

Example 3.2 *Haller [54] proposed the velocity field*

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix} \sin 4t & \cos 4t + 2 & 0 \\ \cos 4t - 2 & -\sin 4t & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}, \tag{3.16}$$

as a purely kinematic benchmark example for vortex criteria. By inspection of the equivalent form (3.6) of proposition 3.1, one immediately finds that (3.16) is actually dynamically consistent, solving the Navier–Stokes equation with $a(t) = \sin 4t$, $b(t) = \cos 4t$, *and* $K = 2$. *More generally, by comparison with (3.6), the linear unsteady velocity field*

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix} \sin\alpha(t-t_0) & \cos\alpha(t-t_0) + K \\ \cos\alpha(t-t_0) - K & -\sin\alpha(t-t_0) \end{pmatrix} \mathbf{x} + \mathbf{f}(t) \tag{3.17}$$

solves the Navier–Stokes equations for any constants α, K *and* t_0, *and any smooth function* $\mathbf{f}(t)$. *We set* $\mathbf{f}(t) \equiv \mathbf{0}$ *for simplicity and pass to a rotating frame via the transformation*

$$\mathbf{x} = \mathbf{Q}(t)\,\mathbf{y}, \qquad \mathbf{Q}(t) = \begin{pmatrix} \cos\frac{\alpha}{2}(t-t_0) & \sin\frac{\alpha}{2}(t-t_0) \\ -\sin\frac{\alpha}{2}(t-t_0) & \cos\frac{\alpha}{2}(t-t_0) \end{pmatrix}.$$

Differentiating the coordinate change with respect to time and using (3.17) gives the transformed equation of motion

$$\dot{\mathbf{y}} = \tilde{\mathbf{u}}(\mathbf{y}) = \begin{pmatrix} 0 & 1 + \left(K - \frac{\alpha}{2}\right) \\ 1 - \left(K - \frac{\alpha}{2}\right) & 0 \end{pmatrix} \mathbf{y} \tag{3.18}$$

for fluid particles in the rotating frame. This transformed velocity field is steady and defines an autonomous linear differential equation for particle motions that is explicitly solvable. The nature of the solutions depends on the eigenvalues $\lambda_{1,2} = \pm\sqrt{1 - \left(K - \frac{\alpha}{2}\right)^2}$ *of the coefficient matrix in (3.17). Specifically, for* $|K - \frac{\alpha}{2}| < 1$, *we have a saddle-type flow with typical solutions growing exponentially, while for* $|K - \frac{\alpha}{2}| > 1$, *we have a center-type flow in which all*

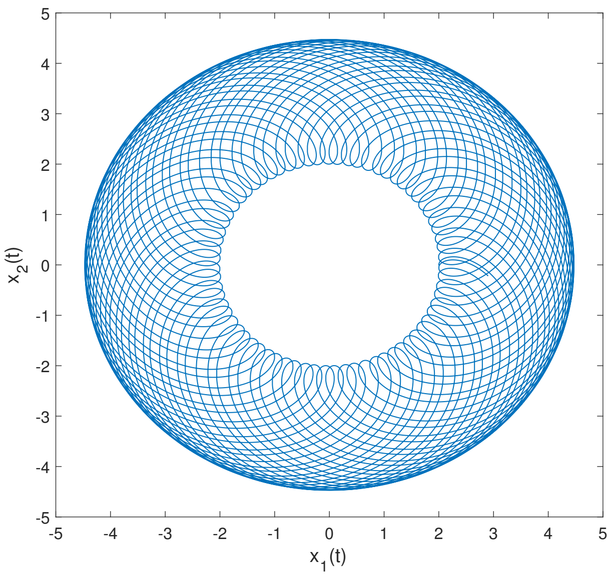

Figure 3.1 – A typical fluid trajectory generated by the linear unsteady velocity field (3.17) for $\alpha = 4$, $K = \frac{1}{2}$, $t_0 = 0$, $t_1 = 200$ for the initial condition $\mathbf{x}_0 = (2, 0)$.

trajectories perform periodic motion. We show an example of such a fluid particle motion transformed back to the original $\mathbf{x}$ *in figure 3.1. Even though this is a flow configuration that most frame-dependent vortex criteria would seek to identify as a vortex, none of the commonly used vortex criteria classifies the original velocity field* (3.17) *as a vortex. At the same time, the same criteria classify the flow* (3.18) *in the rotating frame as a vortex (for details see appendix 3.A). The velocity field family* (3.17) *with* $|K - \frac{\alpha}{2}| > 1$, *therefore, represents a family of false negatives for frame-dependent vortex criteria, complementing the example of a false positive* (3.16) *proposed by Haller [54].*

3.6 Three-dimensional spatially linear Navier–Stokes solutions

The three conditions of theorem 3.1 give the following characterization of three-dimensional spatially linear Navier–Stokes solutions.

Proposition 3.2 *The three-dimensional linear velocity field (3.1) solves the Navier–Stokes equation (3.2) if and only if (3.7) and (3.8) hold.*

Proof Applying the conditions of theorem 3.1 to the symmetric and skew-symmetric components $\mathbf{D}(t)$ and $\mathbf{W}(t)$ of $\mathbf{A}(t)$, it follows that $\mathbf{D}(t)$ must be a continuously differentiable traceless symmetric matrix and, by equation (3.13), that

$$\dot{\mathbf{W}} + \mathbf{D}(t)\mathbf{W}(t) + \mathbf{W}(t)\mathbf{D}(t) = \mathbf{0} \tag{3.19}$$

must hold. Equation (3.19) is a matrix Riccati differential equation with solution (see, e.g., [55])

$$\mathbf{W}(t) = \mathbf{M}(t)\,\mathbf{W}_0\mathbf{M}^T(t),$$

where $\mathbf{M}(t)$ solves the matrix differential equation (3.8). □

Remark 3.2 *As in the two-dimensional case, the set of all matrices $\mathbf{A}(t)$ satisfying the conditions of proposition 3.2 includes the set of all differentiable traceless symmetric matrices and the set of constant skew-symmetric matrices (constant rigid rotations).*

Remark 3.3 [Spatially linear Navier–Stokes solutions with $\mathbf{A}(t) \not\equiv \mathbf{0}$ cannot be Beltrami flows] *A three-dimensional fluid flow with velocity field $\mathbf{u}(\mathbf{x},t)$ is said to be a Beltrami flow if there exists a not-identically-zero scalar function $\lambda(\mathbf{x},t)$ such that*

$$\nabla \times \mathbf{u}(\mathbf{x},t) \equiv \lambda(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t) \tag{3.20}$$

holds for all $\mathbf{x}$ and t. For spatially linear Navier–Stokes solutions, we have $\nabla \times \mathbf{u}(\mathbf{x},t) = \boldsymbol{\omega}(t)$, i.e., the vorticity vector does not depend on $\mathbf{x}$. The curl and gradient of (3.20), therefore, vanish, and give the necessary conditions

$$\nabla \times (\lambda(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)) = \mathbf{0} \tag{3.21}$$

and

$$\lambda(\mathbf{x},t)\,\mathbf{A}(t)+\mathbf{u}(\mathbf{x},t)\,(\nabla\lambda(\mathbf{x},t))^T=\mathbf{0} \tag{3.22}$$

for (3.20) *to hold. Rewriting* (3.21) *as* $\lambda(\mathbf{x},t)\,(\nabla\times\mathbf{u}(\mathbf{x},t))+\nabla(\lambda(\mathbf{x},t))\times\mathbf{u}(\mathbf{x},t)=\mathbf{0}$ *and using the identity* (3.20) *for the first term, we obtain*

$$\lambda^2(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)+\nabla(\lambda(\mathbf{x},t))\times\mathbf{u}(\mathbf{x},t)=\mathbf{0}.$$

Taking the dot product of this last equation with $\mathbf{u}(\mathbf{x},t)$*, the second term vanishes and we are left with*

$$\lambda^2(\mathbf{x},t)\,|\mathbf{u}(\mathbf{x},t)|^2=0.$$

Therefore, $\lambda(\mathbf{x},t)=0$ *must hold for all* $\mathbf{x}$ *and* t *whenever* $\mathbf{u}(\mathbf{x},t)\neq\mathbf{0}$*. In the case* $\mathbf{u}(\mathbf{x},t)=\mathbf{0}$*, however, equation* (3.22) *yields* $\lambda(\mathbf{x},t)\,\mathbf{A}(t)=\mathbf{0}$*. Therefore, whenever* $\mathbf{u}(\mathbf{x},t)=\mathbf{0}$ *holds with* $\mathbf{A}(t)\neq\mathbf{0}$*, we must again have* $\lambda(\mathbf{x},t)=0$ *for* (3.20) *to be satisfied. Therefore,* $\lambda(\mathbf{x},t)$ *must vanish identically, and hence* $\mathbf{u}(\mathbf{x},t)$ *is not a Beltrami flow for* $\mathbf{A}(t)\neq\mathbf{0}$*.*

Example 3.3 [Generalized unsteady impinging jet] *Consider*

$$\mathbf{D}(t)=\begin{pmatrix} g^2(t) & 0 & 0\\ 0 & h^2(t) & 0\\ 0 & 0 & -g^2(t)-h^2(t)\end{pmatrix}, \tag{3.23}$$

where g *and* h *are two continuously differentiable functions, and*

$$\mathbf{W}_0=\begin{pmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}, \tag{3.24}$$

where ω_1*,* ω_2*, and* ω_3 *are all real constants. The general case has been considered by Craik [56], while the special case of* $\omega_1=\omega_2=0$ *has been considered later by Majda and Bertozzi [5].*

The solution $\mathbf{M}(t)$ *to the differential equation* (3.8) *in this case is*

$$\mathbf{M}(t) = \begin{pmatrix} e^{-\int_{t_0}^{t} g^2(s)\,\mathrm{d}s} & 0 & 0 \\ 0 & e^{-\int_{t_0}^{t} h^2(s)\,\mathrm{d}s} & 0 \\ 0 & 0 & e^{\int_{t_0}^{t} g^2(s)+h^2(s)\,\mathrm{d}s} \end{pmatrix}. \tag{3.25}$$

Substituting (3.23)-(3.25) *into formula* (3.7)*, we get that the velocity field*

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} g^2(t) & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & h^2(t) & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & -g^2(t) - h^2(t) \end{pmatrix} \mathbf{x} \tag{3.26}$$

with

$$\begin{aligned} \omega_1(t) &= \omega_1 e^{\int_{t_0}^{t} g^2(s)\,\mathrm{d}s} \\ \omega_2(t) &= \omega_2 e^{\int_{t_0}^{t} h^2(s)\,\mathrm{d}s} \\ \omega_3(t) &= \omega_3 e^{-\int_{t_0}^{t} g^2(s)+h^2(s)\,\mathrm{d}s} \end{aligned}$$

is an exact unsteady solution of the Navier–Stokes equations. It models a rotating jet impinging on the $x_3 = 0$ *plane.*

Example 3.4 [Constant rate-of-strain flows] *Let* $\mathbf{D}_0 \in \mathbb{R}^{3\times 3}$ *be an arbitrary constant traceless symmetric matrix, and let* $\mathbf{W}_0 \in \mathbb{R}^{3\times 3}$ *be an arbitrary constant skew-symmetric matrix. Setting* $\mathbf{D}(t) \equiv \mathbf{D}_0$ *in the differential equation* (3.8) *gives the solution*

$$\mathbf{M}(t) = e^{-\mathbf{D}_0(t-t_0)}.$$

Substitution of $\mathbf{D}(t) \equiv \mathbf{D}_0$, $\mathbf{M}(t)$ *and* $\mathbf{W}_0$ *into formula* (3.7) *gives that*

$$\mathbf{u}(\mathbf{x}, t) = \left[\mathbf{D}_0 + e^{-\mathbf{D}_0(t-t_0)}\mathbf{W}_0 e^{-\mathbf{D}_0(t-t_0)}\right]\mathbf{x} \tag{3.27}$$

is an exact unsteady solution of the Navier–Stokes equation. This solution has a constant rate-of-strain tensor but an unsteady spin tensor.

3.7 Conclusion

We have derived an explicit form for spatially linear unsteady solutions to the Navier–Stokes equations. The fact that linear solutions can systematically be constructed has been known [4–6]. The contribution here is the use of these explicit conditions to construct the explicit form of two- and three-dimensional for spatially linear unsteady solutions to the Navier–Stokes equations. These conditions give a particularly simple form for two-dimensional flows, enabling the verification of dynamical consistency of any linear kinematic flow model by inspection.

The solutions we have identified explicitly solve the Navier–Stokes equation for all Reynolds numbers, including the Euler equation with $\nu = 0$. These solutions are never Beltrami solutions (see remark 3.3) and hence always produce integrable fluid particle motion when $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are chosen to be time-independent constants. Unsteady choices of $\mathbf{A}(t)$ and $\mathbf{f}(t)$, however, can create arbitrarily complex fluid particle motion, including chaotic trajectories.

We have shown that the unsteady linear Navier–Stokes solutions of Dayal and James [53] fall in the general class of explicit solutions we have identified. We have also verified the same property by inspection for the linear unsteady velocity field proposed by Haller [54] as a false positive for frame-dependent vortex criteria. We have extended this example to a larger class of Navier–Stokes solutions that represent false negatives for the same vortex criteria.

In addition, we have constructed further families of nontrivial unsteady Navier–Stokes solutions (generalized unsteady impinging jet and unsteady constant rate-of-strain flow) that can be endowed with a high degree of temporal complexity through the choice of the arbitrary functions in their definitions. (The generalized unsteady impinging jet has been considered before by Craik [56].) Such solutions are expected to be useful as basic unsteady benchmarks for coherent structure detection criteria and numerical schemes.

We are currently working on deriving the explicit form of higher-order polynomial, unsteady velocity field solutions to the Navier-Stokes equations, using the same idea employed in theorem (3.1) for spatially linear solutions. Three-dimensional, higher-order polynomial solutions would be especially interesting since then the nonlinear spatial complexity would allow for chaotic dynamics without the need of explicit time-dependence. The results of this investigation will appear elsewhere.

Appendix

3.A Eulerian vortex criteria

We evaluate several vortex criteria listed below on the Navier–Stokes velocity fields (3.17) and (3.18) from example 3.2 with $\alpha = 4$, $K = \frac{1}{2}$, and $t_0 = 0$. The velocity gradient is $\nabla \mathbf{u}(\mathbf{x}, t) = \mathbf{A}(t)$, and we write $\mathbf{A}(t) = \mathbf{D}(t) + \mathbf{W}(t)$ as per equation (3.11).

The Okubo-Weiss criterion [57, 58] and its three-dimensional counterpart the $\mathcal{Q}$-criterion [59] identify vortices as regions where the vorticity tensor dominates the rate-of-strain tensor:

$$\mathcal{Q} = \frac{1}{2}\left(\|\mathbf{W}(t)\|^2 - \|\mathbf{D}(t)\|^2\right) > 0, \tag{3.28}$$

where $\|\mathbf{W}(t)\| = \sqrt{\mathrm{Tr}\left(\mathbf{W}(t)\,\mathbf{W}^T(t)\right)}$ and $\|\mathbf{D}(t)\| = \sqrt{\mathrm{Tr}\left(\mathbf{D}(t)\,\mathbf{D}^T(t)\right)}$.

The Δ-criterion [60], for three dimensional-flows, identifies vortices as the regions where

$$\Delta = \left(\frac{\mathcal{Q}}{3}\right)^3 + \left(\frac{\det \mathbf{A}(t)}{2}\right)^2 > 0. \tag{3.29}$$

In these regions the velocity gradient $\mathbf{A}(t)$ has complex eigenvalues and so local instantaneous stirring is plausible.

The λ_2-criterion [61] identifies vortices as the regions where

$$\lambda_2\left(\mathbf{D}^2(t)+\mathbf{W}^2(t)\right)<0, \tag{3.30}$$

where $\lambda_2(\mathbf{P}(t))$ denotes the intermediate eigenvalue of the symmetric tensor $\mathbf{P}(t)$. On discarding unsteady straining and viscous effects, the λ_2 criterion guarantees a pressure minimum in a two-dimensional plane for incompressible Navier–Stokes flows.

Finally, the Chakraborty–Balachandar–Adrian criterion [62] identifies a vortex core as the region where $\mathbf{A}(t)$ admits complex eigenvalues $(\lambda_{cr} \pm i\lambda_{ci})$, $\lambda_{ci}>0$, whose real and imaginary parts obey the threshold conditions

$$\begin{aligned} \lambda_{ci} &\geq \epsilon, \\ -\kappa \leq \frac{\lambda_{cr}}{\lambda_{ci}} &\leq \delta, \end{aligned} \tag{3.31}$$

for some positive thresholds ϵ, κ, and δ. The imaginary part λ_{ci} and ratio $\lambda_{cr}/\lambda_{ci}$ are related to the $\mathcal{Q}$ and Δ criteria through the equations

$$\begin{aligned} \mathcal{Q} &= \lambda_{ci}^2\left(1-3\left(\frac{\lambda_{cr}}{\lambda_{ci}}\right)^2\right), \\ \Delta &= \frac{\lambda_{ci}^6}{27}\left(1+9\left(\frac{\lambda_{cr}}{\lambda_{ci}}\right)^2\right)^2. \end{aligned} \tag{3.32}$$

The first threshold condition is a statement on the rate of rotation inside the vortex core. Large (or small) values of λ_{ci} imply long (or short) times to complete a revolution. The second condition is a statement on the orbital compactness of the vortex core. Large (or small) $\lambda_{cr}/\lambda_{ci}$ imply that initially close particles do not (or do) remain neighbors after the time that has elapsed for a complete revolution. The constants κ and δ are threshold parameters for the orbital compactness along the vortex axis and vortex plane, respectively.

3.A.1 Eulerian vortex criteria applied to the unsteady velocity field (3.17)

The relevant matrices and quantities are

$$\mathbf{A}(t) = \begin{pmatrix} \sin 4t & \cos 4t + \frac{1}{2} & 0 \\ \cos 4t - \frac{1}{2} & -\sin 4t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}(t) = \begin{pmatrix} \sin 4t & \cos 4t & 0 \\ \cos 4t & -\sin 4t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{W}(t) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\|\mathbf{W}(t)\| = \sqrt{\operatorname{Tr}\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \frac{1}{\sqrt{2}},$$

$$\|\mathbf{D}(t)\| = \sqrt{\operatorname{Tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \sqrt{2}.$$

The Okubo–Weiss crtierion gives

$$\mathcal{Q} = \frac{1}{2}\left(\|\mathbf{W}(t)\|^2 - \|\mathbf{D}(t)\|^2\right) = \frac{1}{2}\left(\frac{1}{2} - 2\right) = -\frac{3}{4} < 0,$$

which tests negative for vortices. The Δ-criterion gives

$$\Delta = \left(\frac{\mathcal{Q}}{3}\right)^3 + \left(\frac{\det \mathbf{A}(t)}{2}\right)^2 = -\left(\frac{1}{4}\right)^3 < 0,$$

which also tests negative for vortices. The λ_2-criterion gives

$$\lambda_2\left(\mathbf{D}^2(t) + \mathbf{W}^2(t)\right) = \lambda_2\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{3}{4} > 0,$$

which tests negative for vortices. Finally, the Chakraborty–Balachandar–Adrian criterion tests negative since the eigenvalues of $\mathbf{A}(t)$ are real and equal to $\pm\frac{\sqrt{3}}{2}$.

3.A.2 Eulerian vortex criteria applied to the steady velocity field (3.18)

The relevant matrices and quantities are

$$\mathbf{A}(t) = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{W}(t) = \frac{1}{2}\begin{pmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\|\mathbf{W}(t)\| = \sqrt{\operatorname{Tr}\begin{pmatrix} \frac{9}{4} & 0 & 0 \\ 0 & \frac{9}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \frac{3}{\sqrt{2}},$$

$$\|\mathbf{D}(t)\| = \sqrt{\operatorname{Tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \sqrt{2}.$$

The Okubo–Weiss criterion gives

$$\mathcal{Q} = \frac{1}{2}\left(\|\mathbf{W}(t)\|^2 - \|\mathbf{D}(t)\|^2\right) = \frac{1}{2}\left(\frac{9}{2} - 2\right) = \frac{5}{4} > 0,$$

which tests positive for vortices. The Δ-criterion gives

$$\Delta = \left(\frac{\mathcal{Q}}{3}\right)^3 + \left(\frac{\det \mathbf{A}(t)}{2}\right)^2 = \left(\frac{5}{4}\right)^3 > 0,$$

which tests positive for vortices. The λ_2-criterion gives

$$\lambda_2 \begin{pmatrix} -\frac{5}{4} & 0 & 0 \\ 0 & -\frac{5}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{5}{4} < 0,$$

which tests positive for vortices. Finally, the Chakraborty–Balachandar–Adrian criterion tests positive for the threshold parameter $\epsilon \in (0, \frac{\sqrt{5}}{2}]$ since the eigenvalues of $\mathbf{A}(t)$ are complex and equal to $\pm i\frac{\sqrt{5}}{2}$.

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